THESIS WORK: "ADELIC GEOMETRY VIA TOPOS THEORY"

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1. MOTIVATION FROM NUMBER THEORY

Much of the theory-building in Number Theory has been guided by a deep tension: while it is important to treat all the completions of the rationals[1] $\mathbb{Q}$ symmetrically (cf. the Hasse principle), it is also clear that there exist key disanalogies between the p-adics and the reals. The depth of these disanalogies can be measured by the fact that there are many powerful technologies that work well in one setting but not the other.[2] Indeed, as Mazur muses [Maz93]:

> "A major theme in the development of Number Theory has been to try to bring $\mathbb{R}$ somewhat more into line with the p-adic fields; a major mystery is why $\mathbb{R}$ resists this attempt so strenuously."

This leads to a natural question, which will guide the investigations of this thesis.

Question 1. What is the right perspective from which to understand this tension? That is, how can we treat the p-adics and the reals symmetrically whilst also accommodating their differences?

The number theorist is likely to have one of two reactions to Question 1 (and in fact, perhaps both). First, that our understanding of the reals and the p-adics should be guided by the function field analogy. Two, as already alluded to by Mazur, that we should strive to develop tools that work well for both settings. We discuss this in the context of Arakelov intersection theory [Ara74, PR21].

The Function Field Case. Consider a smooth affine curve C over an algebraically closed field k. Then, take the (unique) smooth compactification of C, which adds a finite number of points to yield a smooth projective curve $\overline{C}$. A *divisor* D on $\overline{C}$ is a finite formal linear combination of points on $\overline{C}$

$$D = \sum_{P \in \overline{C}} n_P \cdot P, \qquad n_P \in \mathbb{Z}. \tag{1}$$

In particular, for any non-zero *rational function* f on $\overline{C}$, one can define the divisor

$$(f) = \sum_{P \in \overline{C}} \mathrm{ord}_P(f) \cdot P, \tag{2}$$

where $\mathrm{ord}_P(f)$ denotes the multiplicity of f at P. One can then compute the *degree* of divisor (f) and deduce

$$\deg(f) = \sum_{P \in \overline{C}} \mathrm{ord}_P(f) = 0, \tag{3}$$

a key result that allows us to develop a good intersection theory of divisors.

[1] In fact, the same issue arises for a general number field, but we shall almost always restrict to $\mathbb{Q}$ for simplicity.

[2] One indication of this is that many choose to work with just the p-adics (e.g. via the finite adele ring $\mathbb{A}_{\mathbb{Q}}^{\mathrm{fin}}$) and ignore the reals. See, for instance Balchin-Greenlees' work [BG20] on Adelic Models for tensor-triangulated categories, or Huber's work [Hub91] on the Beilinson-Parshin adeles, where she writes: "We want to stress that at this stage only a generalization of the finite adeles is found. It is not clear what one should take at infinity, or in fact even what the infinite 'places' should be."

The Number Field Case. Consider $\mathbb{Q}$ and the spectrum of the ring of integers $\mathrm{Spec}(\mathbb{Z})$. Notice the non-zero primes $p \in \mathrm{Spec}(\mathbb{Z})$ each corresponds to the p-adic numbers $\mathbb{Q}_p$. To account for $\mathbb{R}$, we formally add to $\mathrm{Spec}(\mathbb{Z})$ the set of complex embeddings $\sigma\colon \mathbb{Q} \hookrightarrow \mathbb{C}$; in which case, this gives a single embedding factoring through $\mathbb{R}$.[3] Denote this enlargement of $\mathrm{Spec}(\mathbb{Z})$ as $\Lambda_{\mathbb{Q}}$, which we shall call *the set of places of* $\mathbb{Q}$. Following standard conventions, we denote the "real prime" adjoined to $\mathrm{Spec}(\mathbb{Z})$ as ∞.

Next, define the *Arakelov divisor* D on $\Lambda_{\mathbb{Q}}$ as the following finite formal linear combination:

$$D = \sum_p n_p \cdot p + \alpha_\infty \cdot \infty, \qquad n_p \in \mathbb{Z},\ \alpha_\infty \in \mathbb{R}, \tag{4}$$

where the first sum runs over the set of non-zero primes in $\mathrm{Spec}(\mathbb{Z})$. As before, given any non-zero rational $f \in \mathbb{Q}$, one can define its Arakelov divisor (f), whose *Arakelov degree* can be computed to give

$$\widehat{\deg}(f) = \sum_{v \in \Lambda_{\mathbb{Q}}} \log |f|_v = 0. \tag{5}$$

Considered side-by-side, the analogy between the two setups becomes clear, but notice the formal nature of the number field case. In the function field case, we added points to the smooth affine curve C by performing a geometric construction on C ("smooth compactification"). By contrast, the number field case starts with a formal abstraction: take the underlying set of $\mathrm{Spec}(\mathbb{Z})$. It is this formal move that allows us to combine the set of primes with the set of complex embeddings (even though they are *a priori* different objects), giving a new set $\Lambda_{\mathbb{Q}}$ which we use to index the summands of the Arakelov divisor.

This style of point-set reasoning ("take the set of ...") is widely accepted in classical mathematics, but here it presents a challenge to our understanding. For one, extending the function field analogy, one would like to regard $\Lambda_{\mathbb{Q}}$ as the compactification of $\mathrm{Spec}(\mathbb{Z})$. But on what grounds? Strictly speaking, $\Lambda_{\mathbb{Q}}$ is just a set of elements with no topology — it is only by analogy that one might regard it as morally being a kind of compactified affine curve. Second, notice that the construction of $\Lambda_{\mathbb{Q}}$ is still guided by an obvious case-split between the p-adics vs. the reals. In fact, as pointed out in [Bak08], Arakelov intersection theory uses very different-looking tools to deal with these two components[4], raising sharp questions about the extent to which Arakelov theory successfully resolves the lack of symmetry between the p-adics and the reals.

2. Connections to Topos Theory

Having provided some number-theoretic context, we now shift gears and discuss the connection to the logical aspects of topos theory. Our main point of leverage is the following structure theorem:

Theorem 2. Every (Grothendieck) topos $\mathcal{E}$ is a classifying topos of some geometric theory $\mathbb{T}_{\mathcal{E}}$. Conversely, every geometric theory $\mathbb{T}$ has a classifying topos $\mathcal{S}[\mathbb{T}]$.

The precise definitions of the relevant terms will be deferred till later. For now, let us just say:

- A *theory* $\mathbb{T}$ is a set of logical axioms that describes structures of interest (e.g. groups, rings etc.);
- *Geometric logic* is a logic that is tailored to reflect topology, e.g. connectives $\wedge$ and $\bigvee$ to match intersection and union of opens. A *geometric theory* is a set of axioms expressed in geometric logic;
- A *model* $M_{\mathbb{T}}$ of a geometric theory $\mathbb{T}$ is a structure satisfying the description expressed by $\mathbb{T}$;
- A *topos* $\mathcal{E}$ is some kind of category satisfying certain nice properties;[5]
- A *classifying topos of* $\mathbb{T}$, denoted $\mathcal{S}[\mathbb{T}]$, is a topos representing the universe of all models of $\mathbb{T}$. In particular, it contains a *generic model* $G_{\mathbb{T}}$, which is generic in the informal sense that it gives a blueprint from which all models $M_{\mathbb{T}}$ of $\mathbb{T}$ can be derived.[6]

[3] In the general case of a number field K, the construction involves adding $[K : \mathbb{Q}]$ many complex embeddings to $\mathrm{Spec}(\mathcal{O}_K)$.

[4] Baker's remark [Bak08] was made in the context of motivating the development of non-Archimedean potential theory, which aims to formulate an Arakelov theory that applies analytic methods from potential theory not only at the Archimedean places but also at the non-Archimedean places too. For details on how this works for curves, see, e.g. [BR10].

[5] Convention: the unqualified term "topos" will always mean a Grothendieck 1-topos, unless stated otherwise. The expert reader may take the phrase "nice properties" to mean Giraud's axiomatic characterisation of a topos.

[6] More precisely: given any $\mathbb{T}$-model $M_{\mathbb{T}}$ living in any topos $\mathcal{E}$, there exists a functor $f^*\colon \mathcal{S}[\mathbb{T}] \to \mathcal{E}$, unique up to isomorphism, such that $f^*(G_{\mathbb{T}}) \cong M_{\mathbb{T}}$ whilst also preserving colimits and finite limits.

This sets up the following question: does there exist a geometric theory $\mathbb{T}_{\text{comp}}$ whose models are the completions of $\mathbb{Q}$ (up to topological equivalence)? Notice if yes, then Theorem 2 gives a classifying topos of $\mathbb{T}_{\text{comp}}$ along with a generic model, which we shall call *the generic completion of* $\mathbb{Q}$.

Why might this be an interesting perspective? We give two natural reasons. First, the generic model $G_{\mathbb{T}}$ of any geometric theory $\mathbb{T}$ is conservative, i.e. given any property ϕ expressible in geometric logic, ϕ holds for $G_{\mathbb{T}}$ iff ϕ holds for all models of $\mathbb{T}$. There are a couple ways to read this in the present context. One interpretation: the generic completion of $\mathbb{Q}$ is a device that allows us to reason about properties that hold for all completions of $\mathbb{Q}$ in a symmetric manner — much like the adele ring $\mathbb{A}_{\mathbb{Q}}$ in classical number theory. Another interpretation: the generic completion of $\mathbb{Q}$ is a construction possessing no other properties besides being a completion of $\mathbb{Q}$. As such, if we wish to calibrate our understanding of the p-adics vs. the reals, it can be helpful to have a well-defined object that distills precisely what their shared similarities are.

The second, and more fundamental, reason is that the topos-theoretic perspective pulls Question 1 away from classical set theory, and opens it up to new tools from logic and category theory. This requires some explanation. To the uninitiated, the existence of serious interactions between number theory and logic may come as a surprise, but this itself is certainly not new. For instance, continuing with the function field analogy, a remarkable transfer theorem was proved by the model theorists back in the 1960s:

Theorem 3 (Ax-Kochen-Eršov Principle [AK65, Er65])**.** As our setup,

- Let $\mathcal{U}$ be a non-principal (= contains all cofinite sets) ultrafilter on the set of primes;
- Let $\prod_p \mathbb{Q}_p/\mathcal{U}$ be the ultraproduct of p-adic fields $\mathbb{Q}_p$;
- Let $\prod_p \mathbb{F}_p((t))/\mathcal{U}$ be the ultraproduct of the fields of formal Laurent series over $\mathbb{F}_p$.

Then, $\prod_p \mathbb{Q}_p/\mathcal{U}$ and $\prod_p \mathbb{F}_p((t))/\mathcal{U}$ are elementarily equivalent.

In plainer terms, the Ax-Kochen-Eršov Principle says: given any first-order logical statement ϕ about valued fields, there exists a finite set C of primes such that ϕ holds for $\mathbb{F}_p((t))$ iff ϕ holds for $\mathbb{Q}_p$ just in case $p \notin C$. As a beautiful application of this result, Ax and Kochen [AK65] proved that every homogeneous polynomial of degree d with more than d^2 variables has a non-trivial solution in $\mathbb{Q}_p$ for all but finitely many primes p. However, while this breakthrough result may be vindicating for the classical logician, its non-constructive aspects makes it problematic for the topos theorist. In particular, notice that the Ax-Kochen-Eršov Principle is formulated using non-principal ultrafilters, whose existence implies a weak form of choice and thus cannot be shown constructively.[7]

This discussion sets up an important organising principle of this thesis. Properly understood, Theorem 2 gives rise to a new understanding of a topos as a so-called "point-free space", which we define below:

Definition 4. A *(point-free) space* is a space X whose points are the models of a geometric theory. A map $f\colon X \to Y$ is defined by a geometric construction of points $f(x) \in Y$ out of points $x \in X$.

This unusual marriage of topology and logic, which we call "point-free topology", differs from the classical perspective in two important ways. One, it challenges the classical notion of a space as a set decorated with a chosen topology. Two, it generalises the classical notion of model as a set decorated with the logical data of relations and/or functions that have been singled out for study. Further details will be explained in due course, but notice that this perspective already gives some indication of how point-free topology systematically pulls our mathematics away from its underlying set theory.

Returning to our original context, what does the point-free perspective mean for Question 1? The methodological upshot: in order to work with models as if they were points of some kind of generalised space (embodied by the topos), we shall need to adhere to a strict regime of constructive mathematics known as *geometric mathematics* [Vic07, Vic14].[8] In practice, "working geometrically" means abandoning many classical tools and principles in exchange for new ones. Unlike the model theorist, we do not have the axiom of choice, and so we shall prefer to work with the generic model of a theory rather than the ultraproducts of

[7]This fact follows from a combination of two results: [HL67] shows that the strongest form of the ultrafilter lemma (= all filters can be extended to ultrafilters) does not imply the standard Axiom of Choice; [Bla77] shows that the weakest form of ultrafilter the lemma (= there exists a non-principal ultrafilter on some set) cannot be proved in ZF set theory.

[8]To work geometrically means to reason using constructions that are preserved by pullback along geometric morphisms between toposes. As will be explained in due course, this essentially means working with constructions/properties built out of finite limits and arbitrary colimits.

its models[9]; and unlike the classical number theorist, we cannot take the underlying set of $\mathrm{Spec}(\mathbb{Z})$ (at least, not without losing geometricity), and so we must find other ways of dealing with the places of $\mathbb{Q}$.

3. Overview of Thesis

Hereafter, the term "space" shall always mean a point-free space (cf. Definition 4) unless stated otherwise. As previously discussed, this thesis will focus on the following test problem:

Problem 5. Construct and describe the classifying topos of completions of $\mathbb{Q}$ (up to equivalence).

Step One: Point-free Real Exponentiation. The first step towards constructing this topos is understanding when a given completion K of $\mathbb{Q}$ is topologically equivalent to another completion K'. Classically, completions of $\mathbb{Q}$ are defined as point-set spaces comprising the Cauchy sequences of $\mathbb{Q}$ with respect to some kind of metric on $\mathbb{Q}$:

$$\begin{aligned} |\cdot|\colon \mathbb{Q} &\longrightarrow [0,\infty) \\ x &\longmapsto |x| \end{aligned} \tag{6}$$

known as an *absolute value*. Given two absolute values $|\cdot|_1, |\cdot|_2$, we can define an equivalence relation $\sim$ where $|\cdot|_1 \sim |\cdot|_2$ iff there exists some $\alpha \in (0,1]$ such that $|x|_1^\alpha = |x|_2$ or $|x|_2^\alpha = |x|_1$ for all $x \in \mathbb{Q}$ such that $x \neq 0$. Such an equivalence class of absolute values is called a *place*, and it turns out that two absolute values belong to the same place iff their completions are topologically equivalent. This reduces an *a priori* topological problem to an algebraic one, except that we shall first need a geometric account of real exponentiation.

Theorem A. There exists an exponentiation map on the Dedekinds

$$\exp\colon (0,\infty)\times\mathbb{R} \to (0,\infty), \tag{7}$$

satisfying the usual exponent laws

$$\begin{aligned} x^{\zeta+\zeta'} &= x^\zeta x^{\zeta'}, \quad x^0 = 1 \\ x^{\zeta\cdot\zeta'} &= (x^\zeta)^{\zeta'}, \quad x^1 = x \\ (xy)^\zeta &= x^\zeta y^\zeta, \quad 1^\zeta = 1. \end{aligned} \tag{8}$$

The result itself is not surprising; the main challenge in the construction are the technical constraints imposed by geometricity. Just as we cannot take the underlying set of $\mathrm{Spec}(\mathbb{Z})$, we may not take the underlying set of the Dedekinds and treat exponentiation as a purely algebraic construction on its elements. Further, exponentiation x^ζ is monotonic in the exponent when $x \in (1,\infty)$ whereas it is antitonic in the exponent when $x \in (0,1)$. This indicates a natural case-splitting on the base, which requires careful justification since, working geometrically, we generally cannot assume the Law of Excluded Middle.

Something interesting that already emerges at this stage are the so-called one-sided reals, which are essentially semi-continuous versions of the usual Dedekinds. Note: while the points of the Dedekinds and one-sided reals more or less coincide classically[10], they are very different entities in geometric mathematics. Here, the one-sided reals serve primarily as computational tools: our general approach involves developing exponentiation for the one-sideds, before lifting the result to the Dedekinds. A similar approach is adopted when developing a geometric account of logarithms:

Theorem B. Fix $b \in (1,\infty)$. We can then define one-sided logarithm maps[11]

$$\log_b\colon \overrightarrow{[0,\infty]} \to \overrightarrow{[-\infty,\infty]} \quad \text{and} \quad \log_b\colon \overleftarrow{[0,\infty]} \to \overleftarrow{[-\infty,\infty]} \tag{9}$$

[9] Why the comparison between ultraproducts of models with the generic model? The short answer: because both constructions, properly understood, lead to representative models of their first-order theories. The case for the generic model is clear given the fact that it is conservative. The case for the ultraproduct construction is more involved — see [Mal19] for details, particularly the discussion on regular ultrapowers and Keisler's Order.

[10] For the sake of argument, let us presently ignore the one-sided infinities.

[11] Convention: intervals of one-sided reals are indicated with an arrow on top, indicating the direction of the Scott topology with respect to the numerical order.

inverse to the corresponding exponentiation maps $b^{(-)}$ on the one-sideds. These combine to yield an isomorphism on the Dedekinds

$$\log_b \colon (0,\infty) \xrightarrow{\sim} (-\infty,\infty). \tag{10}$$

Step Two: Investigation of Absolute Values. The next step in tackling Problem 5 is to construct the topos of absolute values and provide a geometric proof of Ostrowski's Theorem. As before, we shall prove the result for the one-sided reals (in fact, just the upper reals) and the Dedekinds, but now the one-sided reals take on a conceptual significance.

Theorem C (Ostrowski's Theorem for $\mathbb{Z}$)**.** As our setup, denote:

- $\overleftarrow{[av]}$:= The space of absolute values on $\mathbb{Z}$, valued in upper reals.
- $\mathrm{ISpec}(\mathbb{Z})$:= The space of prime ideals of $\mathbb{Z}$.
- $\overleftarrow{[-\infty,1]}$:= The space of upper reals bounded above by 1.

Define

$$\mathfrak{P}_\Lambda := \{(\mathfrak{p},\lambda) \in \mathrm{ISpec}(\mathbb{Z}) \times \overleftarrow{[-\infty,1]} \mid \lambda < 0 \leftrightarrow \exists a \in \mathbb{Z}_{\neq 0}.(a \in \mathfrak{p})\}. \tag{11}$$

Then, we get the following isomorphism of spaces:

$$\overleftarrow{[av]} \cong \mathfrak{P}_\Lambda. \tag{12}$$

Theorem D (Ostrowski's Theorem for $\mathbb{Q}$)**.** As our setup,

- Let $|\cdot|$ be a non-trivial absolute value on $\mathbb{Q}$;
- Let $|\cdot|_\infty$ be the standard Euclidean absolute value, whose completion of $\mathbb{Q}$ yields the reals $\mathbb{R}$;
- Let $|\cdot|_p$ be the standard p-adic absolute value, whose completion of $\mathbb{Q}$ yields the p-adic field $\mathbb{Q}_p$.

Then, one of the following must hold:

(i) $|\cdot| = |\cdot|_\infty^\alpha$ for some $\alpha \in (0,1]$; or
(ii) $|\cdot| = |\cdot|_p^\alpha$ for some $\alpha \in (0,\infty)$ and some prime $p \in \mathbb{N}_+$.

Setting aside the issues of geometricity, let us highlight the differences between Theorems C and D. Theorem D, which is the standard formulation of Ostrowski's Theorem, is a classification result on the absolute values on $\mathbb{Q}$. Since an absolute value $|\cdot|$ on $\mathbb{Q}$ is obviously determined by where it sends the non-zero integers $\mathbb{Z}$, this suggests a natural extension of Ostrowski's Theorem for absolute values defined on $\mathbb{Z}$, which gives Theorem C. Notice that Theorem C as formulated is not just a classification result but also a representation result: not only can we associate any absolute value $|\cdot|$ on $\mathbb{Z}$ to a pair $(p,\lambda) \in \mathfrak{P}_\Lambda$, but this association is also unique (up to unique isomorphism).

There is, however, a deeper point to be made. When defining the theory of absolute values on $\mathbb{Z}$

$$|\cdot| \colon \mathbb{Z} \to \overleftarrow{[0,\infty)}$$

we defined them as multiplicative seminorms valued in the upper reals; on the other hand, we chose to define absolute values on $\mathbb{Q}$ as being valued in the Dedekinds. Geometricity shows this to be canonical. In particular, if we wish to define absolute values valued in upper reals, then we lose the ability to axiomatise positive definiteness and so are forced to consider just the multiplicative seminorms on $\mathbb{Z}$; conversely, if we wish to define absolute values on $\mathbb{Q}$, then they must be valued in Dedekinds instead of the upper reals, which can be shown to give us positive definiteness for free.

The idea of a space whose points correspond to multiplicative seminorms, such as $\overleftarrow{[av]}$, is not new (see e.g. [Ber90]); what *is* new is the tight connection with the upper reals, revealing a subtle interplay between the topology and algebra that was previously hidden. In a slightly more classical interlude, we extend this insight to sharpen a foundational result in Berkovich geometry. In our language,

Theorem E (Berkovich's Disc Theorem)**.** Fix K to be an algebraically closed field complete with respect to a non-Archimedean norm. Define $\mathcal{A}$ to be a ring of convergent power series, i.e.

$$\mathcal{A} := K\{R^{-1}T\} = \left\{ f = \sum_{i=0}^{\infty} c_i T^i \;\middle|\; c_i \in K, \lim_{i\to\infty} |c_i| R^i = 0 \right\}, \tag{13}$$

and define its Berkovich Spectrum $\mathcal{M}(\mathcal{A})$ to be the space of bounded multiplicative seminorms on $\mathcal{A}$.

Then, the space $\mathcal{M}(\mathcal{A})$ is classically equivalent to the space of R-good filters.

There is a hidden surprise here for the expert. Berkovich's original result holds that all points of $\mathcal{M}(\mathcal{A})$ can be described as nested descending sequences of rigid discs *so long as* the norm on K is non-trivial. It is well-known that his argument breaks down when we consider trivially-valued K. However, by using point-free techniques from [Vic05, Vic09], we found the correct modification of rigid discs and were thus able to eliminate the non-triviality hypothesis from Berkovich's result. Not only does this tighten the comparison between the classical Ostrowski's Theorem and Berkovich's Disc Theorem (for reasons we will explain in due course), it also shows that the previous *algebraic* hypothesis of K being non-trivially valued is in fact a *point-set* hypothesis, and is not essential to the underlying mathematics. The surprising aspects of this result hints at the clarifying potential of the point-free techniques (even when applied classically), and motivates a very interesting series of test problems on their interactions with non-Archimedean geometry, which we discuss.

Step Three: Topos of Places of $\mathbb{Q}$. An important payoff for working geometrically is that, leveraging Theorem 2, we have at our disposal a deep collection of structure theorems for toposes, such as descent, that allows us to extract topological information from our logical setup.

This motivates our work here, which investigates the topos of places of $\mathbb{Q}$. Here we explore the question: considered as a point-free space, what do the places of $\mathbb{Q}$ "look" like? A central theme is that while it is clear that the exponentiation of absolute values gives an algebraic action, characterising the point-free spaces quotiented by this action is a subtler issue.

Applying the classification result of Theorem D, we first localise and define the topos of a single non-Archimedean place, denoted $\mathcal{D}$, associated to some prime p. By characterising $\mathcal{D}$ as an appropriate descent topos, we get the following result:

Theorem F. $\mathcal{D} \simeq \mathrm{Set} = \mathcal{S}\{*\}$.

In other words, a single non-Archimedean place corresponds to a singleton $\{*\}$, as one might expect classically. However, here comes the big surprise. When we apply a similar analysis to the topos of the Archimedean (i.e. the real) place, denoted $\mathcal{D}'$, we instead get:

Theorem G. $\mathcal{D}' \simeq \mathcal{S}\overleftarrow{[0,1]}$.

This result overturns a longstanding classical assumption in number theory. Instead of corresponding to a singleton with no intrinsic features (as is assumed in, e.g. Arakelov geometry), Theorem G indicates that the Archimedean place corresponds to a sort of blurred unit interval comprising the upper reals.

As such, our understanding of the mechanics underlying Question 1 has started to shift. At this critical juncture, we are still sorting through the implications of our results, but interesting fragments of the picture have emerged. We first give a topos-theoretic insight on the differences between the non-Archimedean vs. Archimedean place: in our language, whereas $\mathcal{D}$ eliminates all forms of non-trivial forking in its sheaves, upper-bound forking still persists in $\mathcal{D}'$. After which, we identify and discuss a key theme that has been hidden in plain sight in our investigations: namely, the interactions between the connected and the disconnected. This theme turns out to have a surprisingly far reach. On the topos-theoretic side, we discuss how its relation to Theorem F brings into focus an interesting limitation of classifying toposes, raising challenging questions about its intended role in modern applications. On the number-theoretic side, notice that Theorems F and G only give a characterisation of individual places and not of the entire space of places (much less the entire space of completions). In fact, as we discuss, the question of how the Archimedean and non-Archimedean components fit together is surprisingly subtle, and also appears bound up with questions about reconciling the connected with the disconnected. Nonetheless, some very interesting parallels have emerged between our work and Clausen-Scholze's framework of Condensed Mathematics, particularly in regards to the differences between solidity and p-liquidity. This gives us some useful clues on where to start looking for answers.

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