School of Computer Science
University of Birmingham

Adelic Geometry via Topos Theory

(joint work with Steve Vickers)

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Throwback to Como ’18

What this talk is about

I'm going to discuss two basic themes that cross-cut many different areas of mathematics:

1. What kind of info can homotopical data encode?
2. When can we solve a problem by breaking it into smaller pieces?

I'll then discuss how the research project 'Adelic Geometry via Topos Theory' serves as an interesting test problem for illuminating how these two themes interact with each other.

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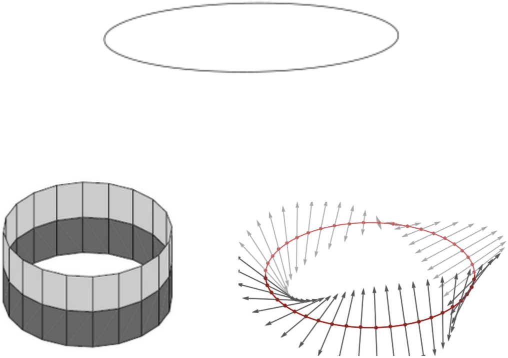

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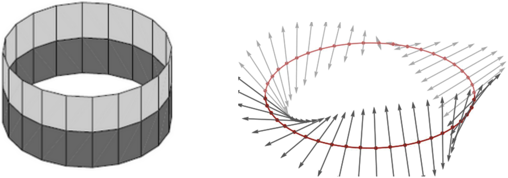

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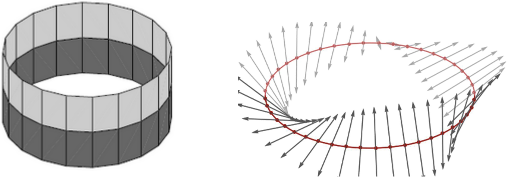

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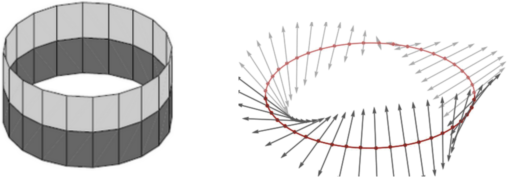

- Question: Are these the only line bundles over S^1 (up to isomorphism)?
- Answer: Yes.
- Why? Exploit the tight relationship between $[S^{k-1}, GL_n(\mathbb{R})]$ and $\mathrm{Vect}_n(S^k)$.

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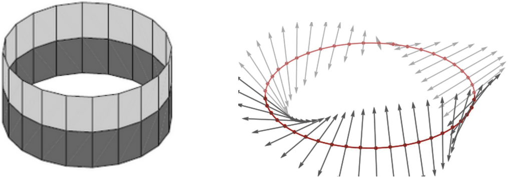


Classification Theorem

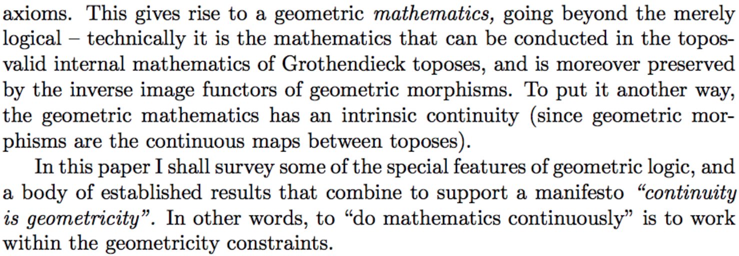
Suppose that X is a paracompact space. Let $\mathrm{Vect}_n(X)$ be the set of isomorphism classes of n-dimensional vector bundles over X. Then the map

$$[X, G_n] \longrightarrow \mathrm{Vect}_n(X)$$

given by $f \longmapsto f^*(\gamma_n)$ is a bijection, where γ_n is the universal bundle.

Continuity = Geometricity

A similar attitude occurs in topos theory in regards to geometric logic:



axioms. This gives rise to a geometric *mathematics*, going beyond the merely logical – technically it is the mathematics that can be conducted in the topos-valid internal mathematics of Grothendieck toposes, and is moreover preserved by the inverse image functors of geometric morphisms. To put it another way, the geometric mathematics has an intrinsic continuity (since geometric morphisms are the continuous maps between toposes).

In this paper I shall survey some of the special features of geometric logic, and a body of established results that combine to support a manifesto *“continuity is geometricity”*. In other words, to “do mathematics continuously” is to work within the geometricity constraints.

— Vickers, ’Continuity and Geometric Logic’

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Geometric Theory

A *geometric theory* is a theory whose (formulae featured in its) axioms are built out of certain logical connectives — i.e. $=$, finite conjunctions $\wedge$, arbitrary (possibly infinite) disjunctions $\bigvee$, and $\exists$.

Example: Theory of Dedekind Reals

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$$L = \{q \in \mathbb{Q} | q < x\}$$

$$R = \{r \in \mathbb{Q} | x < r\}$$

Otherwise known as the left and right Dedekind sections of the real number.

Points of a Topos

Definition

- A *geometric morphism* $f : \mathcal{F} \to \mathcal{E}$ of toposes is a pair of adjoint functors $f_* : \mathcal{F} \to \mathcal{E}$ and $f^* : \mathcal{E} \to \mathcal{F}$, respectively called the *direct image* and the *inverse image* of f, such that the left adjoint f^* preserves finite limits.

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Definition

1. A *global point* of a topos $\mathcal{E}$ is defined as a geometric morphism $\mathrm{Set} \to \mathcal{E}$.
2. A *generalised point* of a topos $\mathcal{E}$ is a geometric morphism $\mathcal{F} \to \mathcal{E}$.

Topos = Generalised Space, Take 2

Definition

The classifying topos of a geometric theory $\mathbb{T}$ is a Grothendieck topos $\mathrm{Set}[\mathbb{T}]$ that classifies the models of $\mathbb{T}$ in Grothendieck toposes, i.e. for any Grothendieck topos $\mathcal{E}$, we have an equivalence of categories:

$$\textbf{Geom}(\mathcal{E}, \mathrm{Set}[\mathbb{T}]) \simeq \mathbb{T}\text{-}\mathrm{mod}(\mathcal{E})$$

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Slogan

Models = points of a topos. In particular, we can reason in terms of the points of the topos (as a generalised space) as opposed to only reasoning in terms of its objects/sheaves (as a category).

Example of Point-wise reasoning

Recall that given a geometric theory $\mathbb{T}$, its classifying topos satisfies the following universal property:

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In particular, letting $\mathbb{R}$ be the propositional theory of Dedekind reals, then we obtain:

$$\textbf{Geom}(\mathrm{Set}[\mathbb{R}], \mathrm{Set}[\mathbb{R}]) \simeq \mathbb{R}\text{-}\mathrm{mod}(\mathrm{Set}[\mathbb{R}])$$

It is well known that given a generic Dedekind real x, one can define $x + x$ geometrically and $x + x$ is also a Dedekind real. That is, $x + x$ is also a $\mathbb{R}$-model in $\mathrm{Set}[\mathbb{R}]$ and this (by the universal property) corresponds to a geometric morphism $\mathrm{Set}[\mathbb{R}] \to \mathrm{Set}[\mathbb{R}]$.

Point-free Topology - A Bird's Eye View

Point-set Topology

- Point = Element of a set
- Space = A set of points, along with a set of opens satisfying some specific axioms.
- Continuous Maps = A function $f : X \to Y$ such that $f^{-1}(U)$ is open for all opens $U \subset Y$

Pointfree Topology

- Point = Model of a geometric theory
- Space = The 'World' in which the point lives with other points (i.e. a Grothendieck topos)
- Continuous Maps = A geometric morphism $f : \mathcal{E} \to \mathcal{F}$ such that $f^* : \mathcal{F} \to \mathcal{E}$ preserves finite limits and small colimits

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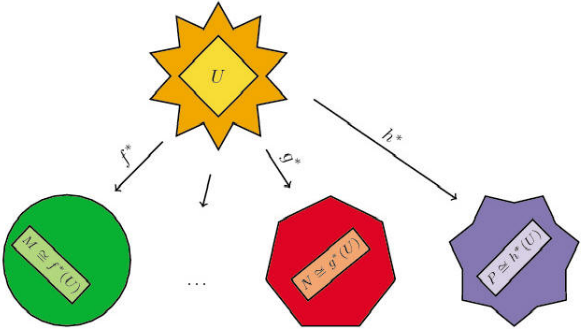
An **important** consequence of this is that any geometric sequent that holds for $U_{\mathbb{T}}$ will hold for all models M of $\mathbb{T}$.

Generic Model



Classifying topos

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- Observation #1: Integer solutions imply real and modulo p solutions (in fact p-adic solutions).
- Observation #2: Real and p-adic solutions are easier to deal with than just integer/rational solutions.

- New Question: Given a polynomial with $\mathbb{Q}$-coefficients, when does knowledge about its $\mathbb{Q}_p$ and $\mathbb{R}$-solutions give us info about its $\mathbb{Q}$-solutions?

Hasse’s Local-Global Principle

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Some property P is true for $\mathbb{Q}$ iff P is true for all the completions of $\mathbb{Q}$.

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Definition of adele ring for $\mathbb{Q}$

The adele ring $\mathbb{A}_{\mathbb{Q}}$ is defined to be the restricted product of all the completions of $\mathbb{Q}$. Morally, the adele ring can be viewed as a device that allows us to reason about all the completions of $\mathbb{Q}$ simultaneously.

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Idea

Instead of asking whether a property simultaneously holds for *all completions* of $\mathbb{Q}$ (which forces us to use complicated algebraic constructions like the adele ring $\mathbb{A}_{\mathbb{Q}}$), what if we asked whether a property holds for the *generic completion* of $\mathbb{Q}$?

Generic Completion

“One weakness in the analogy between the collection of $\{K_s\}_{s \in S}$ for a compact Riemann surface S and the collection $\{\mathbb{Q}_p$, for prime numbers p, and $\mathbb{R}\}$ is that [...] no manner of squinting seems to be able to make $\mathbb{R}$ the least bit mistakeable for any of the p-adic fields, nor are the p-adic fields $\mathbb{Q}_p$ isomorphic for distinct p.

A major theme in the development of Number Theory has been to try to bring $\mathbb{R}$ somewhat more into line with the p-adic fields; *a major mystery is why $\mathbb{R}$ resists this attempt so strenuously.”*

— Mazur, ’Passage from Local to Global in Number Theory’

Investigation begins...

Starting point:

For simplicity, let us assume that our base field is $\mathbb{Q}$. Classically, an absolute value of $\mathbb{Q}$ is a function $|\cdot| : \mathbb{Q} \to \mathbb{R}$ such that for all $x, y \in \mathbb{Q}$:

- $|x| \geq 0$, and $|x| = 0$ iff $x = 0$

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We define a *place* as an equivalence class of absolute values whereby $|\cdot|_1 \sim |\cdot|_2$ if there exists some $\alpha \in (0, 1]$ such that $|\cdot|_1 = |\cdot|_2^{\alpha}$ or $|\cdot|_2 = |\cdot|_1^{\alpha}$.

Classifying Topos of Places of $\mathbb{Q}$

- Intuitively: what does this topos look like?
- The points of this topos would correspond to equivalence classes of absolute values, such that:

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- π is the projection map sending $(|\cdot|, \alpha) \mapsto |\cdot|$
- *ex* is the exponentiation map sending $(|\cdot|, \alpha) \mapsto |\cdot|^{\alpha}$

Pointfree Construction of Real Exponentiation

Ng-Vickers (2021)

GOAL: Construct an exponentiation map

$$\begin{aligned} ex : \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} &\rightarrow \mathbb{R}_{>0} \\ (x, \alpha) &\mapsto x^{\alpha} \end{aligned}$$

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- Adhere to geometric logic
- Corresponds to the classical account of real exponentiation

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- In essence, we would like to 'quotient' the topos [*av*] by an algebraic action – what is the appropriate quotienting construction here?
  - If the action is **invertible**, then use the standard descent construction.
  - If the action is **not invertible**, then use the lax descent construction.

Standard Descent vs. Lax Descent

Consider a 2-truncated simplicial topos $\mathcal{E}_\bullet$:

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- We can obtain a topos **Desc**$(\mathcal{S}(G_\bullet))$, whose objects are pairs (F, θ), where:
  - F is a sheaf on G_0
  - $\theta : d_0^* F \xrightarrow{\sim} d_1^* F$ is a (**necessarily invertible**) action map in $\mathcal{S}(G_1)$ satisfying the unit and cocycle conditions.

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- Important: **No requirement** that θ is **invertible**!

Global vs. Local Picture

- Let $\mathcal{E}_1 = \mathcal{S}[av] \times \mathcal{S}(0,1]$
- Let $\mathcal{E}_2 = \mathcal{S}\big([av] \times (0,1] \times_{[av]} [av] \times (0,1]\big)$

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Ostrowski's Theorem for $\mathbb{Q}$

Every absolute value of $\mathbb{Q}$ is equivalent to a (non-Archimedean) p-adic absolute value $|\cdot|_p$ (for some prime p), or the Archimedean absolute value $|\cdot|_\infty$.

Non-Archimedean Place (for fixed prime *p*)

$$\begin{tikzcd}
(0,1) \times (0,\infty) \arrow[r, bend left, "\pi"] \arrow[r, bend right, "ex"'] & (0,1) \arrow[l, "s"] \arrow[r, dashed] & \mathcal{D}
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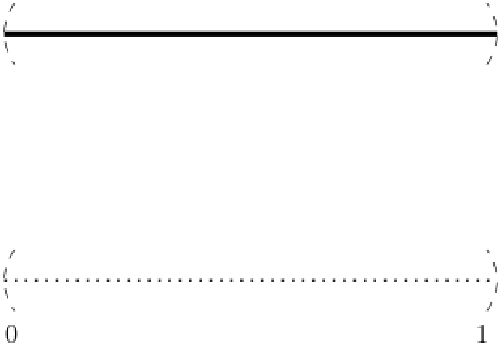
Theorem

$\mathcal{D} \simeq \mathrm{Set}$

Connected components of sheaves

- Crux move:



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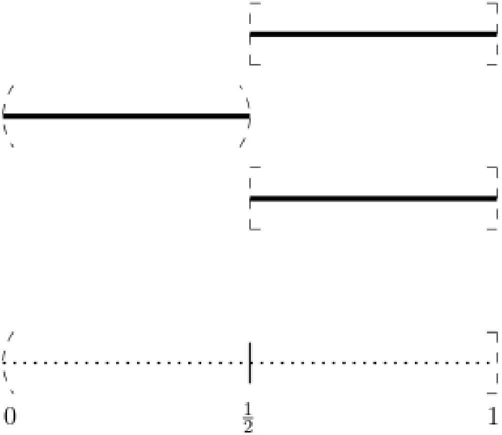
- Space of Arch. absolute values is acted upon by a **monoid** $(0,1]$-action as opposed to a **group** $(0,\infty)$-action.
- Can we play the same game as we did in the Non-Archimedean case?

Forking phenomena



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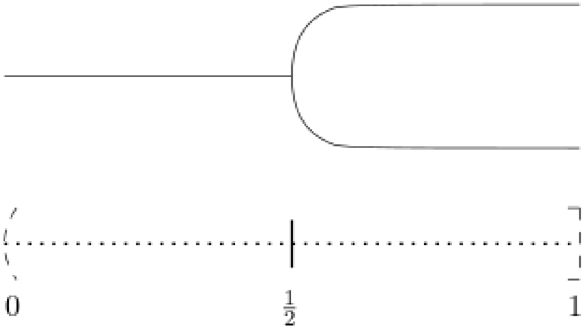

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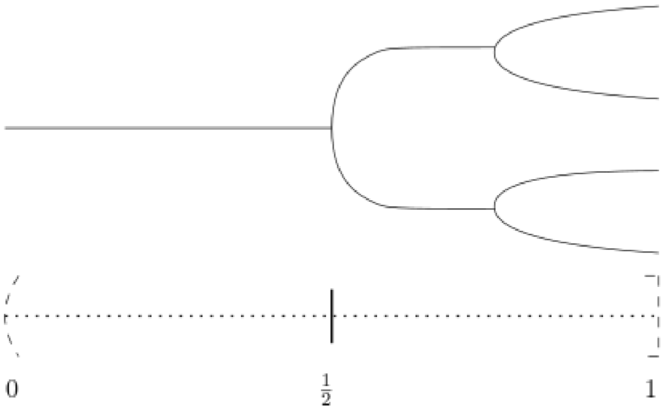

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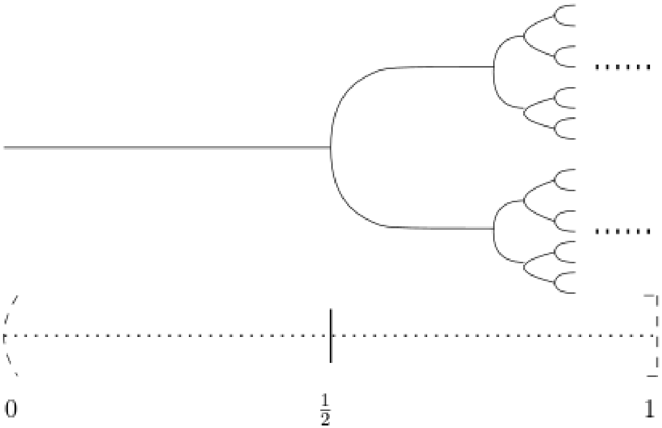

Theory of Upper Reals $\overleftarrow{\mathbb{R}}$

- Recall: the theory of Dedekind reals $\mathbb{R}$ is defined by a list of axioms describing two sets of rationals (L, R) – corresponding to the left and right Dedekind sections respectively.

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- We shall denote $\overleftarrow{[0,1]}$ to be the space of upper reals living between 0 and 1.

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Then f factors uniquely (up to isomorphism) via a map $\overline{f}\colon \overleftarrow{[0,1]} \longrightarrow [\mathbb{O}]$.

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This claim has (essentially) been proved, modulo the fact our proof isn't entirely constructive, so some details still need finessing.

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Observation: Lax Descent and Forking

Recall that the lifting lemma says: for a map $f : \mathbb{Q}_{(0,1]} \to [\mathbb{O}]$ to be lifted to $\bar{f} : \overleftarrow{[0, 1]} \to [\mathbb{O}]$, it should satisfy the colimit condition: $\underset{q<q'}{\underrightarrow{\operatorname{colim}}}\, f(q') \cong f(q)$

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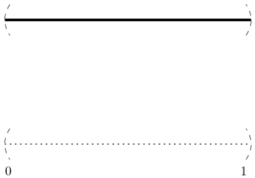
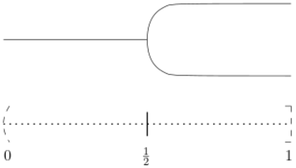


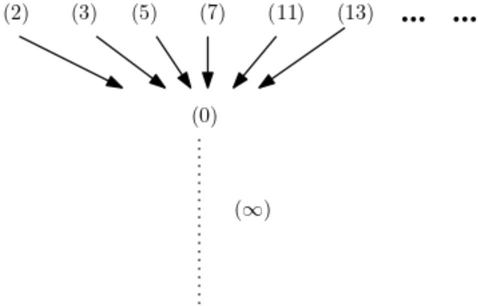


Preliminary Reorientations

Number Theory

Candidate Picture

$\mathcal{D}' \simeq \overleftarrow{[0,1]}$



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- The Arakelov compactification of $\mathrm{Spec}(\mathbb{Z})$ suggests that we add a single point at infinity to $\mathrm{Spec}(\mathbb{Z})$ corresponding to the 'Archimedean prime' . . . our candidate picture suggests that there is some blurring going on at infinity, and that infinity is not just a classical point with no intrinsic structure.

Preliminary Reorientations

Homotopy Theory

Sullivan's Arithmetic Square (a.k.a. 'The Hasse Square'):

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d] & \prod_p \widehat{\mathbb{Z}}_p \arrow[d] \\
\mathbb{Q} \arrow[r] & \mathbb{Q} \otimes_{\mathbb{Z}} \prod_p \widehat{\mathbb{Z}}_p = \mathbb{A}^f_{\mathbb{Q}}
\end{tikzcd}$$

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- Descent and Lax Descent provide a subtle understanding of quotienting of generalised spaces. Further, over certain well-behaved spaces, the forking phenomena of their sheaves gives some kind of topological measure of the action of the lax descent.
- Lax descent also indicates some blurring at infinity in our picture of $\overline{\mathrm{Spec}(\mathbb{Z})}$ — interesting to explore the precise implications of this.

By way of conclusion...

‘One can hope for a very general method of reduction and ‘dévissage’ that transforms a problem of multiple variables into a problem of a single variable, where the difficulty of the original problem is transformed into a problem of working constructively.’

— André Boileau and André Joyal

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