School of Computer Science
University of Birmingham

Adelic Geometry via Geometric Logic

(joint work with Steve Vickers)

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What this talk is about

I’m going to discuss two basic themes:

1. What do homotopical ideas have to do with logic?
2. When can we solve a problem by breaking it into smaller pieces?

I’ll then discuss how the research project ‘Adelic Geometry via Topos Theory’ serves as an interesting test problem for illuminating how these two themes interact with each other.

Point-free Topology - A Bird's Eye View

Point-set Topology

- Point = Element of a set
- Space = A set of points, along with a set of opens satisfying some specific axioms.
- Continuous Maps = A function $f : X \to Y$ that preserves certain structure

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Geometric Theory

A *geometric theory* is a theory whose (formulae featured in its) axioms are built out of certain logical connectives — i.e. $=$, finite conjunctions $\wedge$, arbitrary (possibly infinite) disjunctions $\bigvee$, and $\exists$.

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$$L = \{q \in \mathbb{Q} | q < x\}$$

$$R = \{r \in \mathbb{Q} | x < r\}$$

Otherwise known as the left and right Dedekind sections of the real number.

Example: Theory of Dedekind Reals

The Dedekind sections (L, R) of the real number must satisfy the following (geometric) axioms:

Axioms of $\mathbb{R}$

1. $\exists q \in \mathbb{Q}$ such that $q < x$
2. $q < q' < x \rightarrow q < x$
3. $q < x \rightarrow \exists q' \in \mathbb{Q}$ such that $q < q' < x$
4. $\exists r \in \mathbb{Q}$ such that $x < r$
5. $x < r' < r \rightarrow x < r$.
6. $x < r \rightarrow \exists r' \in \mathbb{Q}$ such that $x < r' < r$
7. $q < x$ and $x < q \rightarrow$ **false**
8. $q < r \rightarrow q < x$ or $x < r$.

Why Points = Models?

Definition

- A *geometric morphism* $f : \mathcal{F} \to \mathcal{E}$ of toposes is a pair of ‘maps’ $f_* : \mathcal{F} \to \mathcal{E}$ and $f^* : \mathcal{E} \to \mathcal{F}$,

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Definition

1. A *global point* of a topos $\mathcal{E}$ is defined as a geometric morphism $\mathrm{Set} \to \mathcal{E}$.
2. A *generalised point* of a topos $\mathcal{E}$ is a geometric morphism $\mathcal{F} \to \mathcal{E}$.

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Definition

The classifying topos of a geometric theory $\mathbb{T}$ is a Grothendieck topos $\mathrm{Set}[\mathbb{T}]$ that classifies the models of $\mathbb{T}$ in Grothendieck toposes, i.e. for any Grothendieck topos $\mathcal{E}$, we have an equivalence of categories:

$$\mathbf{Geom}(\mathcal{E}, \mathrm{Set}[\mathbb{T}]) \simeq \mathbb{T}\text{-}\mathrm{mod}(\mathcal{E})$$

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Slogan

Models = points of a topos. In particular, we can reason in terms of the points of the topos (as a generalised space) as opposed to only reasoning in terms of its objects/sheaves (as a category).

Example of Point-wise reasoning

Recall that given a geometric theory $\mathbb{T}$, its classifying topos satisfies the following universal property:

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In particular, letting $\mathbb{R}$ be the propositional theory of Dedekind reals, then we obtain:

$$\textbf{Geom}(\mathrm{Set}[\mathbb{R}], \mathrm{Set}[\mathbb{R}]) \simeq \mathbb{R}\text{-mod}(\mathrm{Set}[\mathbb{R}])$$

It is well known that given a generic Dedekind real x, one can define $x + x$ geometrically and $x + x$ is also a Dedekind real. That is, $x + x$ is also a $\mathbb{R}$-model in $\mathrm{Set}[\mathbb{R}]$ and this (by the universal property) corresponds to a geometric morphism $\mathrm{Set}[\mathbb{R}] \to \mathrm{Set}[\mathbb{R}]$.

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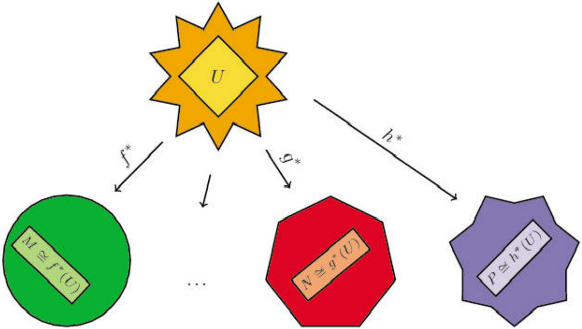
An **important** consequence of this is that any geometric sequent that holds for $U_{\mathbb{T}}$ will hold for all models M of $\mathbb{T}$.

Generic Model



Classifying topos

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- Observation #1: Integer solutions imply real and modulo p solutions (in fact p-adic solutions).
- Observation #2: Real and p-adic solutions are easier to deal with than just integer/rational solutions.

- New Question: Given a polynomial with $\mathbb{Q}$-coefficients, when does knowledge about its $\mathbb{Q}_p$ and $\mathbb{R}$-solutions give us info about its $\mathbb{Q}$-solutions?

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Definition of adele ring for $\mathbb{Q}$

The adele ring $\mathbb{A}_{\mathbb{Q}}$ is defined to be the restricted product of all the completions of $\mathbb{Q}$. Morally, the adele ring can be viewed as a device that allows us to reason about all the completions of $\mathbb{Q}$ simultaneously.

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Idea

Instead of asking whether a property simultaneously holds for *all completions* of $\mathbb{Q}$ (which forces us to use complicated algebraic constructions like the adele ring $\mathbb{A}_{\mathbb{Q}}$), what if we asked whether a property holds for the *generic completion* of $\mathbb{Q}$?

Generic Completion

"One weakness in the analogy between the collection of $\{K_s\}_{s \in S}$ for a compact Riemann surface S and the collection $\{\mathbb{Q}_p$, for prime numbers p, and $\mathbb{R}\}$ is that [...] no manner of squinting seems to be able to make $\mathbb{R}$ the least bit mistakeable for any of the p-adic fields, nor are the p-adic fields $\mathbb{Q}_p$ isomorphic for distinct p.

A major theme in the development of Number Theory has been to try to bring $\mathbb{R}$ somewhat more into line with the *p****-adic fields;*** *a major mystery is why $\mathbb{R}$ resists this attempt so strenuously."*

— Mazur, 'Passage from Local to Global in Number Theory'

Investigation begins...

Starting point:

For simplicity, let us assume that our base field is $\mathbb{Q}$. Classically, an absolute value of $\mathbb{Q}$ is a function $|\cdot| : \mathbb{Q} \to \mathbb{R}$ such that for all $x, y \in \mathbb{Q}$:

- $|x| \geq 0$, and $|x| = 0$ iff $x = 0$

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We define a *place* as an equivalence class of absolute values whereby $|\cdot|_1 \sim |\cdot|_2$ if there exists some $\alpha \in (0, 1]$ such that $|\cdot|_1 = |\cdot|_2^{\alpha}$ or $|\cdot|_2 = |\cdot|_1^{\alpha}$.

Classifying Topos of Places of $\mathbb{Q}$

- Intuitively: what does this topos look like?
- The points of this topos would correspond to equivalence classes of absolute values, such that:

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- π is the projection map sending $(|\cdot|, \alpha) \mapsto |\cdot|$
- *ex* is the exponentiation map sending $(|\cdot|, \alpha) \mapsto |\cdot|^{\alpha}$

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  - Is the notion of (real) exponentiation geometric? Ng-Vickers (2021)
  - What does it mean to quotient by a monoid action vs. group action?

Global vs. Local Picture

Ostrowski's Theorem for $\mathbb{Q}$

Every absolute value of $\mathbb{Q}$ is equivalent to a (non-Archimedean) p-adic absolute value $|\cdot|_p$ (for some prime p), or the Archimedean absolute value $|\cdot|_\infty$.

Non-Archimedean Place (for fixed prime p)

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Theorem

$\mathcal{D} \simeq \mathrm{Set}$

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- So what is $\mathcal{D}'$?

Preliminary Reorientations

Number Theory

Candidate Picture

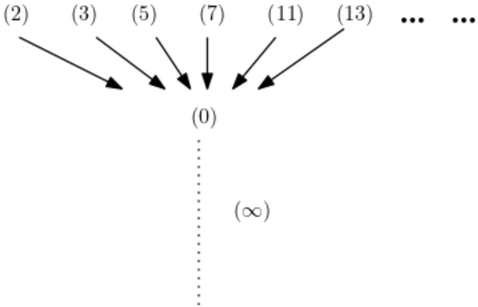
$\mathcal{D}' \simeq \overleftarrow{[0,1]}$

Preliminary Reorientations

Number Theory

Candidate Picture

$\mathcal{D}' \simeq \overleftarrow{[0,1]}$ (the space of ‘upper reals’ between 0 and 1)



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Number Theory

Candidate Picture

$\mathcal{D}' \simeq \overleftarrow{[0,1]}$ (the space of ‘upper reals’ between 0 and 1)

- The Arakelov compactification of $\mathrm{Spec}(\mathbb{Z})$ suggests that we add a single point at infinity to $\mathrm{Spec}(\mathbb{Z})$ corresponding to the ‘Archimedean prime’ . . .

Preliminary Reorientations

Number Theory

Candidate Picture

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- The Arakelov compactification of $\mathrm{Spec}(\mathbb{Z})$ suggests that we add a single point at infinity to $\mathrm{Spec}(\mathbb{Z})$ corresponding to the ‘Archimedean prime’ . . . our candidate picture suggests that there is some blurring going on at infinity, and that infinity is not just a classical point with no intrinsic structure.

Preliminary Reorientations

Homotopy Theory

Sullivan's Arithmetic Square (a.k.a. 'The Hasse Square'):

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d] & \prod_p \widehat{\mathbb{Z}}_p \arrow[d] \\
\mathbb{Q} \arrow[r] & \mathbb{Q} \otimes_{\mathbb{Z}} \prod_p \widehat{\mathbb{Z}}_p = \mathbb{A}^f_{\mathbb{Q}}
\end{tikzcd}$$

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- Theme #1: Viewing toposes as a framework uniting logic and topology
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- Theme #1: Viewing toposes as a framework uniting logic and topology
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- Pulling away from the set theory reveals key insights into the deep nerve connecting topology and algebra.
- Some very interesting indications that there is some blurring at infinity in our picture of $\overline{\mathrm{Spec}(\mathbb{Z})}$ — interesting to explore the precise implications of this.

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