Set Theory vs. Topology

Foundations of Arithmetic & non-Archimedean Geometry

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What this talk is about

Set theory occasionally distorts the foundations of our mathematics, especially in its interactions with topology.

We discuss a couple of examples of this from recent work, partially joint with Steve Vickers.

Foundations in Berkovich Geometry

Not Having Enough Points

Complex algebraic geometry studies complex algebraic varieties[1].

> **Punchline:** These can be regarded as complex manifolds. Hence, they can be studied using tools from complex analysis & differential geometry.

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Question

Can we play the same game for algebraic varieties defined over fields $K \neq \mathbb{C}$?

[1] In fact, let us say: schemes of locally finite type.

A valued field $(K, |\cdot|)$ is called **non-Archimedean** if it satisfies the inequality:

$$|x + y| \leq \max\{|x|, |y|\} \qquad \text{for all } x, y \in K.$$

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- If K is non-Archimedean, trying to naively view an algebraic variety over K as a K-analytic manifold isn't very helpful, since K is totally disconnected.
- Berkovich's solution: Fill K with more points!

Let $(K, |\cdot|)$ be a complete valued field, and $K[T]$ be the polynomial ring.

Multiplicative Seminorm

A **multiplicative seminorm** on D extending the norm of K is a map

$$|\cdot|_x \colon K[T] \to \mathbb{R}_{\geq 0}$$

satisfying the following:

- $|f+g|_x \leq |f|_x + |g|_x \qquad \forall, f, g \in K[T]$
- $|fg|_x = |f|_x|g|_x \qquad \forall, f, g \in K[T]$
- $|a|_x = |a| \qquad \forall a \in K$

Let $(K, |\cdot|)$ be a complete valued field, and $K[T]$ be the polynomial ring.

Berkovich Affine Line

The **Berkovich Affine Line** $\mathbb{A}^1_{\mathrm{Berk}}$ is a space defined as follows:

- *Underlying set of* $\mathbb{A}^1_{\mathrm{Berk}}$ = set of multiplicative seminorms on $K[T]$.
- *Topology of* $\mathbb{A}^1_{\mathrm{Berk}}$ = the Gel'fand topology, i.e. weakest topology such that all maps of the form

$$\begin{aligned} \psi_f \colon \mathbb{A}^1_{\mathrm{Berk}} &\longrightarrow \mathbb{R}_{\geq 0} \\ |\cdot|_x &\longmapsto |f|_x \end{aligned}$$

  are continuous, for any $f \in K[T]$.

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- When $K = \mathbb{C}$, the Gelfand-Mazur Theorem says: $\mathbb{A}^1_{\mathrm{Berk}} \cong \mathbb{C}$.
- When K is non-Archimedean, there are more points in $\mathbb{A}^1_{\mathrm{Berk}}$ than in K.

Classifying Points of Berkovich spaces

Let $(K, |\cdot|)$ be a complete non-Arch. valued field that is algebraically closed. A rigid disc is a subset $D_r(k) \subset K$ of the form

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Berkovich's Classification Theorem

Suppose K is non-trivially valued. <u>Then</u> every point $|\cdot|_x \in \mathbb{A}^1_{\mathrm{Berk}}$ corresponds to a nested sequence of discs

$$D_{r_1}(k_1) \supseteq D_{r_2}(k_2) \supseteq \ldots \tag{1}$$

in the sense that

$$|\cdot|_x = \lim_{n\to\infty} |\cdot|_{D_{r_i}(k_i)} \tag{2}$$

where $|\cdot|_{D_r(k)}$ is the multiplicative seminorm canonically associated to $D_r(k)$.

The same construction and result holds for other rings as well. Here's another important example:

- Let $(K, |\cdot|)$ be a complete non-Arch. field that is algebraically closed.
- Denote $\mathcal{A} := K\{R^{-1}T\}$ to be ring of power series converging in radius R.
- Denote $\mathcal{M}(\mathcal{A})$ to be the analogous space of multiplicative seminorms on $\mathcal{A}$.

$\sum a_i T^i$ s.t.

$\lim_{i \to} |a_i| R^i \to 0$

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Berkovich's Classification Theorem

Suppose K is non-trivially valued. Then, every point $|\cdot|_x \in \mathcal{M}(\mathcal{A})$ is approximated by a nested descending sequence of discs

$$D_{r_1}(k_1) \supseteq D_{r_2}(k_2) \supseteq \ldots \tag{3}$$

in the same sense as before.

On the hypothesis of “non-trivially valued”

The space of multiplicative seminorms is still well-defined even when K is trivially valued.[1]

$|k| = 1 \ \forall k \neq 0$

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On the hypothesis of "non-trivially valued"

The space of multiplicative seminorms is still well-defined even when K is trivially valued.[1] In fact, Berkovich [Ber90] gives an explicit description of these spaces, depending on whether the radius of convergence $R < 1$ or $R \geq 1$.

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On the hypothesis of “non-trivially valued”

- $D_r(k) = \{ b \mid |k-b| \le r \}$
- $|k| = 1 \quad \forall k \neq 0$.

Question: So why assume K to be non-trivially valued?

Obs: $b \neq k$

$\Rightarrow |b-k| = 1$

$D_{\frac{1}{3}}(k) = D_{\frac{1}{2}}(k)$

$\Rightarrow \forall r < 1, \; D_r(k) = \{k\}$

*

On the hypothesis of "non-trivially valued"

Question: So why assume K to be non-trivially valued?

> *The second assumption [that K is non-trivially valued] is necessary [...] if the norm on K is trivial, then there are too few discs.*
>
> — **Jonsson [Jon15]**

Perspective from Point-free Topology

Let us redefine the notion of rigid discs:

Formal Ball

Denote:

- $K_R := \{k \in K \mid |k| \leq R\}$ for some positive real $R > 0$
- Q_+ to be the set of positive rationals.

A **formal ball** is an element $(k, q) \in K_R \times Q_+$. We shall represent this more suggestively as $B_q(k)$. In particular, we write:

$$B_{q'}(k') \subseteq B_q(k) \text{ just in case } |k - k'| < q \wedge q' \leq q.$$

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Key Observation #1: Unlike rigid discs, the radius of formal balls are well-defined, i.e. $B_{q'}(k) = B_q(k')$ implies $q' = q$.

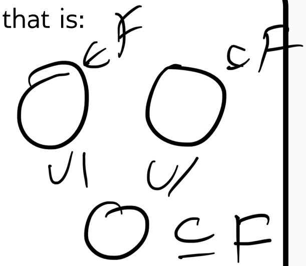
Perspective from Point-free Topology

Also, instead of working with nested sequences of rigid discs, let us consider:

R-good Filter

A **filter** $\mathcal{F}$ of formal balls is an inhabited subset of $K_R \times Q_+$ that is:

- Upward closed w.r.t $\subseteq$
- Closed under pairwise intersections.



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- For any $k \in K_R$, and $q \in Q_+$ such that $R < q$, $B_q(k) \in \mathcal{F}$.
- If $B_q(k) \in \mathcal{F}$, there exists $B_{q'}(k') \in \mathcal{F}$ such that $q' < q$.

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Key Observation #2: Given an R-good filter $\mathcal{F}$, define $\text{rad}_{\mathcal{F}} := \inf\{q \mid B_q(k) \in \mathcal{F}\}$ to be its *radius*. Notice $0 \leq \text{rad}_{\mathcal{F}} \leq R$.

A Sharper Classification Theorem

Theorem (N.)

Setup:

- $(K, |\cdot|)$ is a complete non-Arch field that is algebraically closed – in particular, we allow K to be trivially-valued.
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Slogan: The algebraic hypothesis of being non-trivially valued is in fact a point-set hypothesis.

We can now give new (and shorter) proofs of familiar charaterisations of Berkovich spectra:

$\mathcal{B}_q(\mathcal{A}) \cong K_R \times \mathbb{Q}_f$

$K_R = \{k \mid |k| \le R\}$

Figure 1: LHS: $\mathcal{M}(\mathcal{A})$ when $R < 1$, RHS: $\mathcal{M}(\mathcal{A})$ when $R \geq 1$.

If $R < 1$, $K_R = \{0\}$ #

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1) The use of formal balls reflect the localic perspective that it is the **opens** that are the basic units for defining a space.
2) A topos can be regarded as a generalised space whose points are models of a geometric theory: sometimes these points are algebraic widgets (e.g. groups, rings etc.), sometimes these points are completely prime filters, sometimes both.

The View from Topos Theory

On Logic & Topology

> *Model theory rarely deals directly with topology; the great exception is the theory of o-minimal structures, where the topology arises naturally from an ordered structure.*
>
> — **E. Hrushovski and F. Loeser [HL16]**

> *While geometric logic can be treated as just another logic, it is an unusual one. [...] To put it another way, the geometric mathematics has an intrinsic continuity.*
>
> — **S. Vickers [Vic14]**

What is a space?

Point-set Topology

- Point = An element of a set
- Space = A set of points, along with a collection of subsets satisfying specific properties ("opens of a topology").

Point-free Topology

- Point = A model of a geometric theory
- Space = The 'World' in which all models of the theory live ($\approx$ a topos)

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Geometric Logic

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- **Formula:** Let $\vec{x}$ be a finite vector of variables, each with a given sort. A *geometric formula* in context $\vec{x}$ is a formula built up using symbols from Σ via the following logical connectives: $=$, $\top$ (true), $\wedge$ (**finite** conjunction), $\bigvee$ (**arbitrary** disjunction), $\exists$.

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- **Theory:** A *geometric theory over* Σ is a set of axioms of the form

$$\forall \vec{x}.(\phi \to \psi),$$

where ϕ and ψ are geometric formulae.

Differences with classical logic

- Absence of negation $\neg$
- Allows for arbitrary (possibly infinite) disjunction

Special case: Propositional Theory

Suppose Σ is just a set of propositional symbols (in particular, no sorts).

- Geometric formulae are constructed from these symbols using $\top$, $\wedge$, $\bigvee$.
- A geometric theory over Σ is a set of axioms of the form $\phi \to \psi$.

Localic Space

Recall the following perspective from point-free topology.

- Point = A model of a geometric theory
- Space = The 'World' in which all models of the theory live

If the geometric theory is propositional, we call the corresponding space a **localic space**.

(q, r)

The propositional theory $T_{\mathbb{R}}$ with propositional symbols $P_{q,r}$ (with $q, r \in \mathbb{Q}$, the rationals) and the axioms:

- $P_{q,r} \wedge P_{q',r'} \longleftrightarrow \bigvee\{P_{s,t} | \max(q, q') < s < t < \min(r, r')\}$
- $\top \longrightarrow \bigvee\{P_{q-\epsilon,q+\epsilon} | q \in \mathbb{Q}\}$ for any $0 < \epsilon \in \mathbb{Q}$.

$(q, r) \cap (q', r') = \emptyset$

$q < r < q' < r'$

$(q, r) \cap (q', r') = (q', r)$

Theory of Upper Reals

Consider a subset $R \subset \mathbb{Q}$. For suggestiveness, write "$R < r$" whenever $r \in R$.

Suppose R is subject to the axiom:

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Remark: Morally speaking, an upper real R corresponds to the right Dedekind section of a real. More precisely, upper reals are *classically* equivalent the the usual Dedekind reals[2], but *intuitionistically* they are different.

[2] At least, once we ignore the infinities.

The Language of Filters

Question: Given a point $x \in X$ in some space X, how does the family of open neighbourhoods containing x look like from its POV?

Filter

For I an infinite set, $\mathcal{F} \subset \mathcal{P}(I)$ is a **filter** on I when:

(i) $A \subseteq B \subseteq I$ and $A \in \mathcal{F}$ implies $B \in \mathcal{F}$.

(ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$

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- Call $\mathcal{F}$ a **completely prime filter** whenever

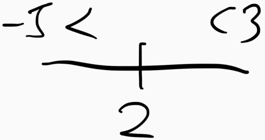
$$\bigcup_{k \in K} A_k \in \mathcal{F} \implies \exists j \in K \text{ s.t. } A_j \in \mathcal{F}$$

  where K is an arbitrary (possibly infinite) indexing set.
- Call $\mathcal{F}$ an **ultrafilter** if it has an opinion on all subsets of I:

  For any subset $S \subseteq I$, either $S \in \mathcal{F}$ or $I \setminus S \in \mathcal{F}$ (but not both).

Reals as Prime Filters vs. Ultrafilters

- Dedekind Real: (L, R)



type = { max. consistent set of formulas }.

ℓ_i "$< a$"$_{a\in\mathbb{Q}}$

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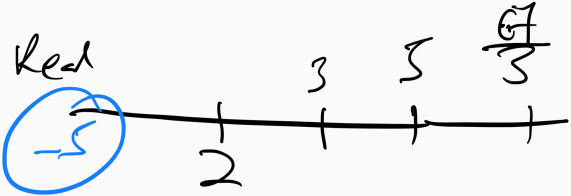
- Upper Real



Question

Given a polynomial with $\mathbb{Q}$-coefficients, say

$$X^n + Y^n + Z^n = 0 \qquad (n > 2),$$

does it have $\mathbb{Q}$-solutions iff it has solutions over all the p-adics $\mathbb{Q}_p$ and reals $\mathbb{R}$?

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Answer: Sometimes.

- **Hasse-Minkowski Theorem:** Quadratic forms have $\mathbb{Q}$-solutions iff they have solutions over all completions of $\mathbb{Q}$.
- **Counter-Examples:**
  - Lind-Reichardt: $2Y^2 = X^4 - 17Z^4$
  - Selmer: $3X^3 + 4Y^3 + 5Z^3 = 0$

Reconciling two worlds

"One weakness in the analogy between the collection of $\{K_s\}_{s \in S}$ for a compact Riemann surface S and the collection $\{\mathbb{Q}_p$, for prime numbers p, and $\mathbb{R}\}$ is that [...] ***no manner of squinting seems to be able to make $\mathbb{R}$ the least bit mistakeable for any of the p-adic fields****, nor are the p-adic fields $\mathbb{Q}_p$ isomorphic for distinct p.*

A major theme in the development of Number Theory has been to try to bring $\mathbb{R}$ somewhat more into line with the p-adic fields; *a major mystery is why $\mathbb{R}$ resists this attempt so strenuously."*

— Barry Mazur [Maz93]

We would still like a device for reasoning about properties satisfied by all completions of $\mathbb{Q}$ simultaneously.

Arakelov Geometry

Consider the 1-point compactification of $\mathrm{Spec}(\mathbb{Z})$: each prime p in $\mathrm{Spec}(\mathbb{Z})$ corresponds to $\mathbb{Q}_p$, and a single point ∞ at infinity corresponding to $\mathbb{R}$.

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Local-Global Principle

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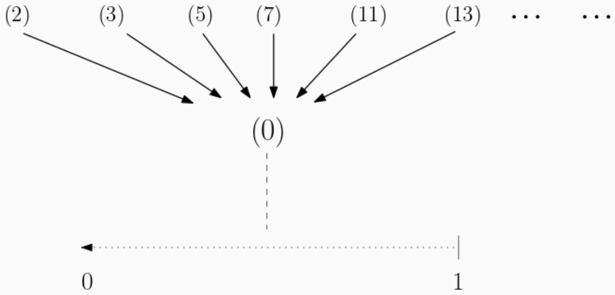
- $p \longleftrightarrow$ A prime ideal of $\mathbb{Z}$
- $\infty \longleftrightarrow$ wtf ???

Theorem (N.-Vickers)

(I) Any Non-Archimedean place of $\mathbb{Q}$ corresponds to a singleton $\{*\}$.

(II) The Archimedean place of $\mathbb{Q}$ corresponds to $\overleftarrow{[0,1]}$, i.e. the space of upper reals between 0 and 1.



Equivalent Characterisations of $\mathbb{A}^1_{\mathrm{Berk}}$

Assume that K is a nice field[a]. We may characterise $\mathbb{A}^1_{\mathrm{Berk}}$ equivalently as

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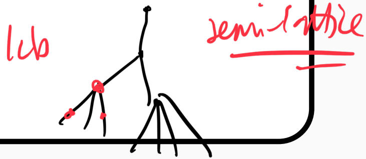
(iv) **Potential Theory:** A profinite $\mathbb{R}$-tree.



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Are these sketches of the same non-Archimedean elephant?

Theorem (Hrushovski-Loeser [HL16])

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The Berkovich analytification of a quasi-projective variety is locally contractible and has the homotopy type of a finite simplicial complex.

Theorem (van der Put-Schneider [vdPS95])

Let X be an affinoid space over a complete non-Arch. non-trivial field K.

- The space of **prime filters** $\mathcal{P}(X)$ is isomorphic to the **adic space** of valuations on the affinoid algebra $\mathcal{O}(X)$ of X.
- The maximal Hausdorff quotient space $\mathcal{M}(X)$ comprising the **ultrafilters** on X corresponds to the **Berkovich space** (= valuations of rank 1).

By way of conclusion

Question: what is the role of set theory in topology?

1) **Berkovich Geometry:** As stated, Berkovich's Classification theorem for $K\{R^{-1}T\}$ fails for trivially valued K due to essentially point-set reasons.

2) **Arithmetic Geometry:** Classically, the Archimedean place of $\mathbb{Q}$ is treated as a singleton because of the assumption that points correspond to elements of a set.

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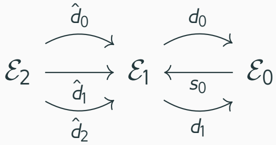
2) **Arithmetic Geometry:** Classically, the Archimedean place of $\mathbb{Q}$ is treated as a singleton because of the assumption that points correspond to elements of a set.

In different ways, we used the point-free perspective to pull these problems away from the underlying set theory. Both results indicate a particular loss of information within the classical setting. The implications involve but go beyond concerns about constructivity, revealing a deep nerve connecting topology & algebra that had previously been obscured.

Lax Descent Construction. Consider a 2-truncated simplicial topos $\mathcal{E}_\bullet$:



We can obtain a category **LDesc**$(\mathcal{E}_\bullet)$ as the coinserter for the diagram (subject to the usual descent conditions). Its objects are pairs (F, θ), where:

- F is a sheaf of $\mathcal{E}_0$
- $\theta : d_0^* F \to d_1^* F$ is a morphism in $\mathcal{E}_1$ satisfying the unit and cocycle conditions.

Important: Unlike the standard descent topos, **no requirement** that θ is **isomorphism**!

Methodological challenge

As stated, the descent construction treats the topos as a category of objects, rather than a generalised space of models. To reformulate this in the point-free language, we decided to regard the sheaves as étale bundles, which keeps the connection with the point-free perspective.

Proof of Theorem

To prove the theorem, the basic plan of attack is to construct two functors

$$F \colon \mathcal{D}' \rightleftarrows \mathcal{S}[0,1] \colon G$$

where $\mathcal{D}'$ is the lax descent topos, and prove that they are inverse. The mathematical devil lies in the details.

- G is induced by the fact that there exists a natural map from Dedekinds to upper reals defined by forgetting the left Dedekind section.
- F is trickier, and involved constructing a technical lifting lemma, and showing that sheaves over $(0,1]$ restricted to the rationals $\mathbb{Q}_{(0,1]}$ (that obey the lax descent conditions) also satisfy the conditions of the lemma.

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