School of Computer Science
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Adelic Geometry via Topos Theory

(joint work with Steve Vickers)

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What this talk is about

I'm going to discuss two basic themes:

1. In what sense are Grothendieck toposes generalised spaces?
2. When can we solve a problem by breaking it into smaller pieces?

I'll then discuss how the research project 'Adelic Geometry via Topos Theory' serves as an interesting test problem for illuminating how these two themes interact with each other.

Point-free Topology - A Bird's Eye View

Point-set Topology

- Point = Element of a set
- Space = A set of points, along with a set of opens satisfying some specific axioms.
- Continuous Maps = A function $f : X \to Y$ that preserves certain structure

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Example: Theory of Dedekind Reals

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$$L = \{q \in \mathbb{Q} | q < x\}$$

$$R = \{r \in \mathbb{Q} | x < r\}$$

Otherwise known as the left and right Dedekind sections of the real number.

Topos = Generalised Space

Theorem

Every Grothendieck topos 'classifies' the models of some theory $\mathbb{T}$, and every geometric theory $\mathbb{T}$ is 'classified' by some Grothendieck topos.

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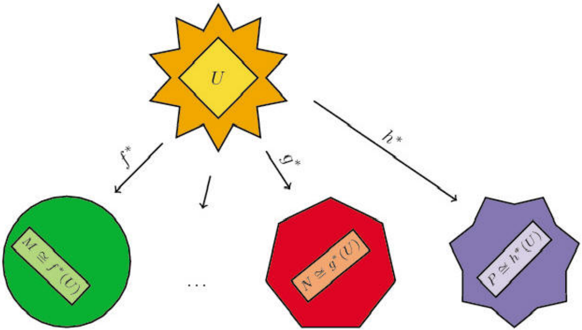
Slogan

Models = points of a topos. In particular, we can reason in terms of the points of the topos (as a generalised space) as opposed to just reasoning in terms of its objects/sheaves (as a category).

Generic Model



Classifying topos

‘Divide-and-Conquer’

$$X^n + Y^n + Z^n = 0 \qquad (n > 2)$$

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- Observation #2: Real and p-adic solutions are easier to deal with than just integer/rational solutions.

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- Observation #2: Real and p-adic solutions are easier to deal with than just integer/rational solutions.

- New Question: Given a polynomial with $\mathbb{Q}$-coefficients, when does knowledge about its $\mathbb{Q}_p$ and $\mathbb{R}$-solutions give us info about its $\mathbb{Q}$-solutions?

Hasse's Local-Global Principle

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Definition of adele ring for $\mathbb{Q}$

The adele ring $\mathbb{A}_{\mathbb{Q}}$ is defined to be the restricted product of all the completions of $\mathbb{Q}$. Morally, the adele ring can be viewed as a device that allows us to reason about all the completions of $\mathbb{Q}$ simultaneously.

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Idea

Instead of asking whether a property simultaneously holds for *all completions* of $\mathbb{Q}$ (which forces us to use complicated algebraic constructions like the adele ring $\mathbb{A}_{\mathbb{Q}}$), what if we asked whether a property holds for the *generic completion* of $\mathbb{Q}$?

Investigation begins...

Starting point:

For simplicity, let us assume that our base field is $\mathbb{Q}$. Classically, an absolute value of $\mathbb{Q}$ is a function $|\cdot| : \mathbb{Q} \to \mathbb{R}$ such that for all $x, y \in \mathbb{Q}$:

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We define a *place* as an equivalence class of absolute values whereby $|\cdot|_1 \sim |\cdot|_2$ if there exists some $\alpha \in (0, 1]$ such that $|\cdot|_1 = |\cdot|_2^\alpha$ or $|\cdot|_2 = |\cdot|_1^\alpha$.

Classifying Topos of Places of $\mathbb{Q}$

- Intuitively: what does this topos look like?
- The points of this topos would correspond to equivalence classes of absolute values, such that:

    1. $|\cdot|^{\alpha} \sim |\cdot|$

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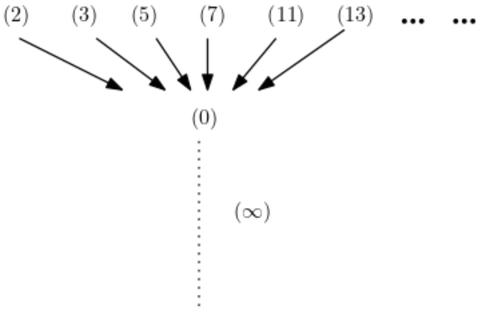
- In essence, we would like to 'quotient' the topos [*av*] by an algebraic action – two questions:
  - Is the notion of (real) exponentiation geometric? Ng-Vickers (2021)
  - What does it mean to quotient by a monoid action vs. group action?

Preliminary Reorientations



Candidate Picture

$\mathcal{D}' \simeq \overleftarrow{[0,1]}$

- The Arakelov compactification of $\mathrm{Spec}(\mathbb{Z})$ suggests that we add a single point at infinity to $\mathrm{Spec}(\mathbb{Z})$ corresponding to the 'Archimedean prime' ...

Preliminary Reorientations

Candidate Picture

$\mathcal{D}' \simeq \overleftarrow{[0,1]}$

- The Arakelov compactification of $\mathrm{Spec}(\mathbb{Z})$ suggests that we add a single point at infinity to $\mathrm{Spec}(\mathbb{Z})$ corresponding to the 'Archimedean prime' ... our candidate picture suggests that there is some blurring going on at infinity, and that infinity is not just a classical point with no intrinsic structure.

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- Theme #1: Viewing toposes as a framework uniting logic and topology
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By way of conclusion...

- Theme #1: Viewing toposes as a framework uniting logic and topology
- Theme #2: Local-Global issues, and its connections to Theme #1 via generic reasoning

- Pulling away from the set theory reveals key insights into the deep nerve connecting topology and algebra (via descent theory)
- Some very interesting indications that there is some blurring at infinity in our picture of $\overline{\mathrm{Spec}(\mathbb{Z})}$ — interesting to explore the precise implications of this.

By way of conclusion...

‘One can hope for a very general method of reduction and ‘dévissage’ that transforms a problem of multiple variables into a problem of a single variable, where the difficulty of the original problem is transformed into a problem of working constructively.’

— André Boileau and André Joyal

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