School of Computer Science
University of Birmingham

Adelic Geometry via Geometric Logic

(joint work with Steve Vickers)

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December 7, 2021

What this talk is about

I'm going to discuss two basic themes:

1. What does topology have to do with logic?
2. When can we solve a problem by breaking it into smaller pieces?

I'll then discuss how the research project 'Adelic Geometry via Topos Theory' serves as an interesting test problem for illuminating how these two themes interact with each other.

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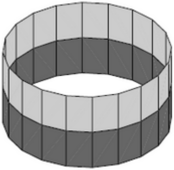

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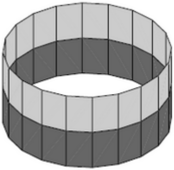

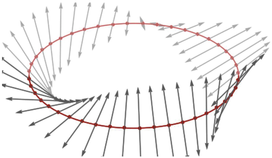

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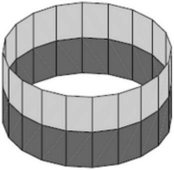

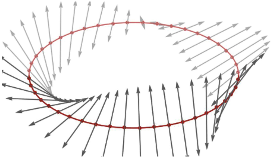

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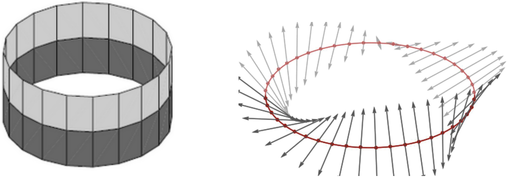

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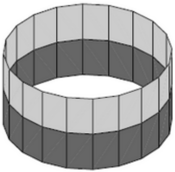

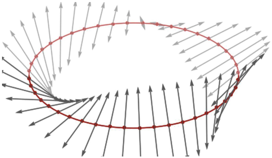

- Question: Are these the only line bundles over S^1 (up to isomorphism)?
- Answer: Yes.
- Why? Exploit the tight relationship between $[S^{k-1}, GL_n(\mathbb{R})]$ and $\mathrm{Vect}_n(S^k)$.

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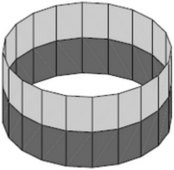

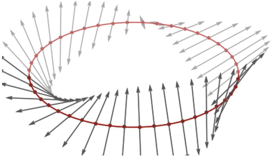

Classification Theorem

Suppose that X is a paracompact space. Then there exists a bijection

$$[X, G_n] \cong \mathrm{Vect}_n(X)$$

where G_n is the classifying space.

Continuity = Geometricity

A similar attitude occurs in topos theory in regards to geometric logic:

> In this paper I shall survey some of the special features of geometric logic, and a body of established results that combine to support a manifesto *"continuity is geometricity"*. In other words, to "do mathematics continuously" is to work within the geometricity constraints.

— Vickers, 'Continuity and Geometric Logic'

Geometric Logic

Let Σ be a (many-sorted) first-order signature. It comprises:

- A set of *sorts*.
- A set of *function symbols*, with finite arity.
- A set of *relation symbols*, again with finite arity.

Geometric Theory

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A *geometric theory* $\mathbb{T}$ over Σ is a theory whose (formulae featured in its) axioms are built out of certain logical connectives — i.e. $=$, $\top$, **finite** conjunctions $\wedge$, **arbitrary** (possibly infinite) disjunctions $\bigvee$, and $\exists$.

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Special Case: Propositional Theory

In the special case where Σ has **no** sorts, then we call a geometric theory $\mathbb{T}$ over Σ *propositional* —

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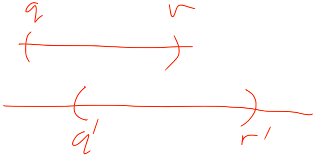
- Classical (finitary) logic: Dedekind cuts arise as *types* of the theory of dense linear orders on $(\mathbb{Q}, <)$.
- Geometric logic: Dedekind reals arise as *models* of a geometric theory.

Geometric Theory of Reals



The propositional theory $T_{\mathbb{R}}$ with propositional symbols $P_{q,r}$ (with $q, r \in \mathbb{Q}$, the rationals) and the axioms:

- $P_{q,r} \wedge P_{q',r'} \leftrightarrow \bigvee\{P_{s,t} | \max(q, q') < s < t < \min(r, r')\}$
- $\top \rightarrow \bigvee\{P_{q-\epsilon,q+\epsilon} | q \in \mathbb{Q}\}$ for any $0 < \epsilon \in \mathbb{Q}$.

Types and Models as Filters

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Filter

For I an infinite set, $\mathcal{F} \subset \mathcal{P}(I)$ is a *filter* on I when:

(i) $A \subseteq B \subseteq I$ and $A \in \mathcal{F}$ implies $B \in \mathcal{F}$.

(ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$

(iii) $\emptyset \notin \mathcal{F}$.

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  The models of $\mathbb{T}$ are the **completely prime filters** of $L_{\mathbb{T}}$.

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Question: What kind of properties should $L, R \subset \mathbb{Q}$ have so that they represent Dedekind sections? Can these properties be formulated via geometric logic?

Theory of Upper Reals

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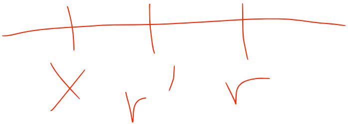
Axioms

1. $x < r' < r \to x < r$.
2. $x < r \to \exists r' \in \mathbb{Q}$ such that $x < r' < r$



Geometric Theory of Dedekind Reals

The Dedekind sections (L, R) of the real number must satisfy the following (geometric) axioms:

Axioms of $\mathbb{R}$

1. $\exists q \in \mathbb{Q}$ such that $q < x$
2. $q < q' < x \rightarrow q < x$
3. $q < x \rightarrow \exists q' \in \mathbb{Q}$ such that $q < q' < x$
4. $\exists r \in \mathbb{Q}$ such that $x < r$
5. $x < r' < r \rightarrow x < r$.
6. $x < r \rightarrow \exists r' \in \mathbb{Q}$ such that $x < r' < r$
7. $q < x$ and $x < q \rightarrow$ **false**
8. $q < r \rightarrow q < x$ or $x < r$.

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Topos = Generalised Space

Toposes are originally defined as *categories*. In what sense then are toposes generalised spaces?

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Definition

1. A *global point* of a topos $\mathcal{E}$ is defined as a geometric morphism $\mathrm{Set} \to \mathcal{E}$.
2. A *generalised point* of a topos $\mathcal{E}$ is a geometric morphism $\mathcal{F} \to \mathcal{E}$.

Topos = Generalised Space

Definition

The classifying topos of a geometric theory $\mathbb{T}$ is a Grothendieck topos $\mathrm{Set}[\mathbb{T}]$ that classifies the models of $\mathbb{T}$ in Grothendieck toposes, i.e. for any Grothendieck topos $\mathcal{E}$, we have an equivalence of categories:

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Slogan

Models = Points of a Topos.

Point-free Topology - A Bird's Eye View

Point-set Topology

- Point = Element of a set
- Space = A set of points, along with a collection of opens satisfying some specific axioms.
- Continuous Maps = A function $f : X \to Y$ such that $f^{-1}(U)$ is open for all opens $U \subset Y$

Pointfree Topology

- Point = Model of a geometric theory
- Space = The 'World' in which the point lives with other points (i.e. a Grothendieck topos)
- Continuous Maps = A geometric morphism $f : \mathcal{E} \to \mathcal{F}$ such that $f^* : \mathcal{F} \to \mathcal{E}$ preserves finite limits and small colimits

The Abstract vs. The Concrete

*"This tension between an **abstract definition** and a **concrete construction** is addressed in both Category Theory and Model Theory.*

*Category Theory is directed at the removal of the importance of a concrete construction. It provides a language to compare different concrecte constructions and in addition **provides a very new way to construct objects** [...] On the other hand, Model theory is concentrated on the gap between an abstract definition and a concrete construction."*

— Kazhdan, Lecture Notes in Motivic Integration.

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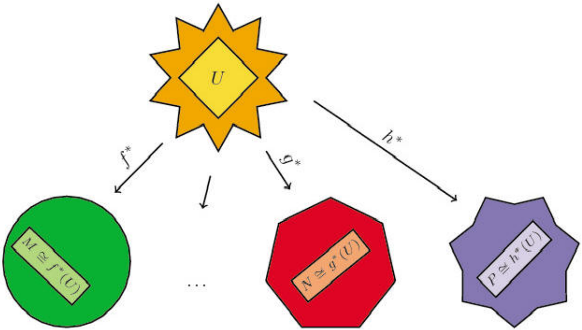
An **important** consequence of this is that any geometric sequent that holds for $U_{\mathbb{T}}$ will hold for all models M of $\mathbb{T}$.

Generic Model



Classifying topos

‘Divide-and-Conquer’

$$X^n + Y^n + Z^n = 0 \qquad (n > 2)$$

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a, b, c $\quad a^n + b^n + c^n = 0$?

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- Observation #2: Real and p-adic solutions are easier to deal with than just integer/rational solutions.

- New Question: Given a polynomial with $\mathbb{Q}$-coefficients, when does knowledge about its $\mathbb{Q}_p$ and $\mathbb{R}$-solutions give us info about its $\mathbb{Q}$-solutions?

Hasse's Local-Global Principle

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Some property P is true for $\mathbb{Q}$ iff P is true for all the completions of $\mathbb{Q}$.

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Definition of adele ring for $\mathbb{Q}$

The adele ring $\mathbb{A}_{\mathbb{Q}}$ is defined to be the restricted product of all the completions of $\mathbb{Q}$. Morally, the adele ring can be viewed as a device that allows us to reason about all the completions of $\mathbb{Q}$ simultaneously.

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Idea

Instead of asking whether a property simultaneously holds for *all completions* of $\mathbb{Q}$, what if we asked whether a property holds for the *generic completion* of $\mathbb{Q}$?

Generic Completion

"One weakness in the analogy between the collection of $\{K_s\}_{s \in S}$ for a compact Riemann surface S and the collection $\{\mathbb{Q}_p$, for prime numbers p, and $\mathbb{R}\}$ is that [...] no manner of squinting seems to be able to make $\mathbb{R}$ the least bit mistake-able for any of the p-adic fields, nor are the p-adic fields $\mathbb{Q}_p$ isomorphic for distinct p.

A major theme in the development of Number Theory has been to try to bring $\mathbb{R}$ somewhat more into line with the p-adic fields; *a major mystery is why $\mathbb{R}$ resists this attempt so strenuously."*

— Mazur, 'Passage from Local to Global in Number Theory'

Investigation begins...

Starting point:

For simplicity, let us assume that our base field is $\mathbb{Q}$. Classically, an absolute value of $\mathbb{Q}$ is a function $|\cdot| : \mathbb{Q} \to \mathbb{R}$ such that for all $x, y \in \mathbb{Q}$:

- $|x| \geq 0$, and $|x| = 0$ iff $x = 0$

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We define a *place* as an equivalence class of absolute values whereby $|\cdot|_1 \sim |\cdot|_2$ if there exists some $\alpha \in (0, 1]$ such that $|\cdot|_1 = |\cdot|_2^\alpha$ or $|\cdot|_2 = |\cdot|_1^\alpha$.

Classifying Topos of Places of $\mathbb{Q}$

- Intuitively: what does this topos look like?
- The points of this topos would correspond to equivalence classes of absolute values, such that:

  1. $|\cdot|^{\alpha} \sim |\cdot|$

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- π is the projection map sending $(|\cdot|, \alpha) \mapsto |\cdot|$
- *ex* is the exponentiation map sending $(|\cdot|, \alpha) \mapsto |\cdot|^\alpha$

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- In essence, we would like to 'quotient' the topos [*av*] by an algebraic action – two questions:

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  - Is the notion of (real) exponentiation geometric? Ng-Vickers (2021)
  - What does it mean to quotient by a monoid action vs. group action?

Ostrowski's Theorem for $\mathbb{Q}$

Every absolute value of $\mathbb{Q}$ is equivalent to a (non-Archimedean) p-adic absolute value $|\cdot|_p$ (for some prime p), or the Archimedean absolute value $|\cdot|_\infty$.

Non-Archimedean Place (for fixed prime p)

$$\begin{tikzcd} {[av_{NA}]} \times (0, \infty) \arrow[r, bend left, "\pi"] \arrow[r, bend right, "ex"'] & {[av_{NA}]} \arrow[l, "s"] \arrow[r, dashed] & \mathcal{D} \end{tikzcd}$$

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Theorem

$\mathcal{D} \simeq \mathrm{Set}$

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- Can we play the same game as we did in the Non-Archimedean case? Answer: No!

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$$
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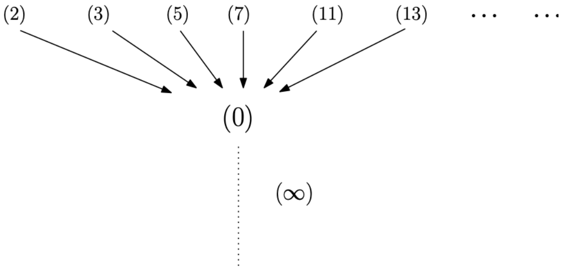
- Can we play the same game as we did in the Non-Archimedean case? Answer: No! (The topos $\mathcal{D}'$ has non-trivial forking in its sheaves)
- So what is $\mathcal{D}'$?

Candidate Picture

$\mathcal{D}' \simeq \overleftarrow{[0,1]}$

Candidate Picture

$\mathcal{D}' \simeq \overleftarrow{[0,1]}$ (the space of 'upper reals' between 0 and 1)



Candidate Picture

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- The Arakelov compactification of $\mathrm{Spec}(\mathbb{Z})$ suggests that we add a single point at infinity to $\mathrm{Spec}(\mathbb{Z})$ corresponding to the 'Archimedean prime' ...

Candidate Picture

$\mathcal{D}' \simeq \overleftarrow{[0,1]}$ (the space of Upper Reals between 0 and 1)

- The Arakelov compactification of $\mathrm{Spec}(\mathbb{Z})$ suggests that we add a single point at infinity to $\mathrm{Spec}(\mathbb{Z})$ corresponding to the 'Archimedean prime' ... our candidate picture suggests that there is some blurring going on at infinity, and that infinity is not just a classical point with no intrinsic structure.

Sullivan's Arithmetic Square (a.k.a. 'The Hasse Square'):

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] & \prod_p \widehat{\mathbb{Z}}_p \arrow[d] \\ \mathbb{Q} \arrow[r] & \mathbb{Q} \otimes_{\mathbb{Z}} \prod_p \widehat{\mathbb{Z}}_p = \mathbb{A}^f_{\mathbb{Q}} \end{tikzcd}$$

. . . . . . Investigations into K-theoretic adeles, augmenting the Arithmetic Square such that it includes $\mathbb{R}$ (joint work in progress with Scott Balchin).

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By way of conclusion...

- Theme #1: Viewing toposes as a framework uniting logic and topology
- Theme #2: Local-Global issues, and its connections to Theme #1 via generic reasoning

- Pulling away from the set theory reveals key insights into the deep nerve connecting topology and algebra.
- Some very interesting indications that there is some blurring at infinity in our picture of $\overline{\mathrm{Spec}(\mathbb{Z})}$ — interesting to explore the precise implications of this.

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