School of Computer Science
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Point-free Construction of Positive Real Exponentiation

(joint work with Steve Vickers)

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Quote du Jour

'Nobody would ever want to do a point-free construction of real exponentiation - it would be too painful.' - Thomas Streicher to Steve Vickers.

Contents

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- Point = Model of a geometric theory
- Space = The 'World' in which the point lives with other points (i.e. a Grothendieck topos)
- Continuous Maps = A geometric morphism $f : \mathcal{E} \to \mathcal{F}$ such that $f^* : \mathcal{F} \to \mathcal{E}$ preserves finite limits and small colimits

Geometric Theory

Let Σ be a set of symbols.

- We build logical statements (i.e. '*geometric formula*') from the symbols in Σ using $=$, finite conjunctions $\wedge$, arbitrary (possibly infinite) disjunctions $\bigvee$, and $\exists$.

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- We then define *geometric axioms* of the form $\phi \to \psi$, where ϕ and ψ are geometric formulae.
- A *geometric theory* over Σ is a set of axioms of the form $\phi \to \psi$, where ϕ and ψ are geometric formulae.

As an example, we now define the geometric theory of Dedekind reals, which we denote $\mathbb{R}$. A model x of $\mathbb{R}$ will is a Dedekind real number, which will be represented by two sets of rationals (L, R), whereby:

$$L = \{q \in \mathbb{Q} | q < x\}$$

$$R = \{r \in \mathbb{Q} | x < r\}$$

Otherwise known as the left and right Dedekind sections of the real number.

The Dedekind sections (L, R) of the real number must satisfy the following (geometric) axioms:

Axioms of $\mathbb{R}$

1. $\exists q \in \mathbb{Q}$ such that $q < x$
2. $q < q' < x \rightarrow q < x$
3. $q < x \rightarrow \exists q' \in \mathbb{Q}$ such that $q < q' < x$
4. $\exists r \in \mathbb{Q}$ such that $x < r$
5. $x < r' < r \rightarrow x < r$.
6. $x < r \rightarrow \exists r' \in \mathbb{Q}$ such that $x < r' < r$
7. $q < x$ and $x < q \rightarrow$ **false**
8. $q < r \rightarrow q < x$ or $x < r$.

An important class of geometric theories:

Propositional Geometric Theory

Let Σ be a set of propositional symbols (in particular, no sorts). *Geometric formulae* over Σ are constructed from the symbols in Σ using $\top$ (true), $\wedge$ and arbitrary disjunctions $\bigvee$.

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Remark #1

Strong analogy between the algebraic structure of propositional geometric formulae and the lattice of open sets of a topological space, otherwise known as a *locale*.

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Remark #2

CONSTRUCTIVE! But also interestingly, the locales associated to propositional theories retain important results of classical topology that often **fail** in constructivist point-set approach (e.g. deMorgan's Laws, $a \vee (X - a) = X$, etc.)

Locales vs. Grothendieck Toposes

Rrecall: A propositional geometric theory can be associated to a (point-free) space known as a *locale*.

More generally, any geometric theory can be associated to a *Grothendieck topos*, which is a special category that can be viewed as a generalisation of locales, and thus a generalised point-free space.

Models as Points of Generalised Spaces

A *theory* can be viewed as an axiomatic description of mathematical structures (e.g. the theory of groups); a *model* is an object that 'satisfies' these axioms.

One may view the models of a geometric theory as the 'points' of the topos.

Definition

1. A *geometric morphism* $f : \mathcal{F} \to \mathcal{E}$ of toposes is a pair of adjoint functors $f_* : \mathcal{F} \to \mathcal{E}$ and $f^* : \mathcal{E} \to \mathcal{F}$, respectively called the *direct image* and the *inverse image* of f, such that the left adjoint f^* preserves finite limits.

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Definition

1. A *global point* of a topos $\mathcal{E}$ is defined as a geometric morphism Set$\to \mathcal{E}$.
2. A *generalised point* of a topos $\mathcal{E}$ is a geometric morphism $\mathcal{F} \to \mathcal{E}$.

We can now make precise the sense in which the points of a topos are the models of the theory it classifies:

Definition

The classifying topos of a geometric theory $\mathbb{T}$ is a Grothendieck topos $\mathit{Set}[\mathbb{T}]$ that classifies the models of $\mathbb{T}$ in Grothendieck toposes, i.e. for any Grothendieck topos $\mathcal{E}$, we have an equivalence of categories:

$$\mathbf{Geom}(\mathcal{E}, \mathit{Set}[\mathbb{T}]) \simeq \mathbb{T}\text{-}\mathit{mod}(\mathcal{E})$$

natural in $\mathcal{E}$, where $\mathbf{Geom}(\mathcal{E}, \mathit{Set}[\mathbb{T}])$ is the category of geometric morphisms from $\mathcal{E}$ to $\mathit{Set}[\mathbb{T}]$ and $\mathbb{T}\text{-}\mathit{mod}(\mathcal{E})$ is the category of models of $\mathbb{T}$ in $\mathcal{E}$.

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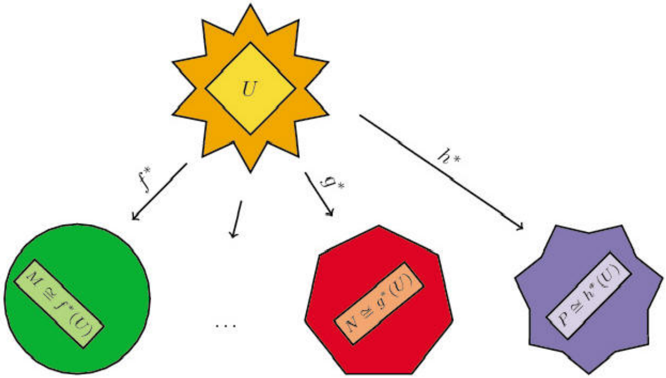
Theorem

Every Grothendieck topos is a classifying topos of some geometric theory $\mathbb{T}$, and every geometric theory $\mathbb{T}$ has a classifying topos.

Fact

There exists something known as the *generic model* $U_{\mathbb{T}}$ in every classifying topos, which possesses the universal property that that any model M in a Grothendieck topos $\mathcal{E}$ can be obtained, up to isomorphism, as a pullback $f^*(U_{\mathbb{T}})$ of the model $U_{\mathbb{T}}$ along the inverse image f^* of a unique geometric morphism $f : \mathcal{E} \to \mathit{Set}[\mathbb{T}]$.

An **important** consequence of this is that any geometric sequent that holds for $U_{\mathbb{T}}$ will hold for all models M of $\mathbb{T}$.



Classifying topos

Excerpt from Steve Vickers' 'Continuity and Geometric Logic':

axioms. This gives rise to a geometric *mathematics*, going beyond the merely logical – technically it is the mathematics that can be conducted in the topos-valid internal mathematics of Grothendieck toposes, and is moreover preserved by the inverse image functors of geometric morphisms. To put it another way, the geometric mathematics has an intrinsic continuity (since geometric morphisms are the continuous maps between toposes).

In this paper I shall survey some of the special features of geometric logic, and a body of established results that combine to support a manifesto *"continuity is geometricity"*. In other words, to "do mathematics continuously" is to work within the geometricity constraints.

Propositional Theory of Dedekind Reals

The Dedekind sections (L, R) of the real number must satisfy the following (geometric) axioms:

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1. $\exists q \in \mathbb{Q}$ such that $q < x$
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7. $q < x$ and $x < q \to$ **false**
8. $q < r \to q < x$ or $x < r$.

We’ve mentioned how mathematics that adheres to geometric logic possesses an abstract form of continuity (since it would be preserved by the inverse image functor of geometric morphisms).

The final axiom of $\mathbb{R}$, often known as locatedness, allows us to express the more familiar notion of continuity, by axiomatising the analytic notion of limits:

If $q < r$, then either $q < x$ or $x < r$.

Indeed (see, e.g. Maietti-Vickers) locatedness is equivalent to:

If (L, R) is a Dedekind real, and $0 < \epsilon \in \mathbb{Q}$, then we can find $q \in L$ and $r \in R$ such that $r - q < \epsilon$.

Propositional Theory of Dedekind Reals

Example of Continuity = Geometricity

Recall that given a geometric theory $\mathbb{T}$, its classifying topos satisfies the following universal property:

$$\mathbf{Geom}(\mathcal{E}, \mathrm{Set}[\mathbb{T}]) \simeq \mathbb{T}\text{-}\mathit{mod}(\mathcal{E})$$

In particular, letting $\mathbb{R}$ be the propositional theory of Dedekind reals, then we obtain:

$$\mathbf{Geom}(\mathrm{Set}[\mathbb{R}], \mathrm{Set}[\mathbb{R}]) \simeq \mathbb{R}\text{-}\mathit{mod}(\mathrm{Set}[\mathbb{R}])$$

It is well known that given a generic Dedekind real x, one can define $x + x$ geometrically and $x + x$ is also a Dedekind real. That is, $x + x$ is also a $\mathbb{R}$-model in Set[$\mathbb{R}$] and this (by the universal property) corresponds to a geometric morphism $\mathrm{Set}[\mathbb{R}] \to \mathrm{Set}[\mathbb{R}]$.

Pointfree Construction of Positive Real Exponentiation

GOAL: Construct an exponentiation map

$$\mathcal{F} : \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$$
$$(x, \alpha) \mapsto x^{\alpha}$$

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- Adhere to geometric logic
- Corresponds to the classical account of positive real exponentiation

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3. Define $x^{\frac{1}{b}}$ for $x \in \mathbb{R}_{\geq 0}$ and $b \in \mathbb{N}$. Then, define x^q for $x \in \mathbb{R}_{\geq 0}$ and q a positive rational (since $q = \frac{a}{b}$ for some $a, b \in \mathbb{N}$).

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4. Define x^α for $x \in \mathbb{R}_{\geq 0}$, and $\alpha \in \mathbb{R}_{>0}$

Pointfree Construction of Positive Real Exponentiation

Seems simple enough, but there were some technical challenges in implementing this strategy:

Challenges

1. Some foundational aspects of point-free analysis have not been worked out, so this needed developing before we could proceed.
2. Proving these constructions satisfies the locatedness axiom whenever x is a (non-negative) Dedekind real, especially in Step 4.
3. Being forced to do case-splitting in Step 4, which had to be justified by delicate gluing arguments.

Step 4: Defining x^α (The general case)

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Motivating Question

Suppose we have shown that x^q is a (non-negative) Dedekind real, for all positive rationals q, and all non-negative Dedekind reals x. How would we define x^α then, for α a positive real?

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Idea from classical analysis

Given some positive *real* α, and for q positive *rational*, $q \to \alpha$, $x^q \to x^\alpha$.

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Johnstone's axiom for $\mathbb{R}$

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So what is x^α?

We would like to define (geometrically) that x^α is the pair of Dedekind sections (L, R) approximated by x^q from above and below, and that as q gets infinitesimally close to α, x^q gets infinitesimally close to x^α.

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Roughly speaking, suppose we have $|\alpha - q| < \epsilon$. How do we determine (constructively) a bound for $|x^\alpha - x^q|$?

Problem #2

In addition, there are also antitone and monotone problems.
For instance, if $x \in (0, 1]$, then for positive rationals $q < r$, we have that $x^r < x^q$ whereas if $x \in (1, \infty)$ then we have $x^q < x^r$.

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3. Glue these constructions together to get a continuous function.

Justifying that all this works geometrically requires a lot of technical effort. There are a couple of interesting themes that I'd like to mention though:

Theme #1: Nice properties of pointfree topology are nice

Recall that one of the attractive features of pointfree topology is that we have constructive analogues of results from classical topology (e.g. $a \vee (X - a) = X$). As it turns out, these play an important role in justifying why we are justified in splitting the non-negative Dedekind reals into the different parts.

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Recall that one of the attractive features of pointfree topology is that we have constructive analogues of results from classical topology (e.g. $a \vee (X - a) = X$)). As it turns out, these play an important role in justifying why we are justified in splitting the non-negative Dedekind reals into the different parts.

Theme #2: Nice properties of $\mathbb{Q}$ are very nice

It is well-known that $<$ is decidable on $\mathbb{Q}$, and virtually all of the classic results of $\mathbb{Q}$ (e.g. the Archimedean property) can be geometrically justified.

As it turns out, these nice properties of $\mathbb{Q}$ can be used to constrain the more opaque (i.e. second-order) behaviour of the Dedekind reals, and was particularly useful when proving that our constructions satisfied the locatedness axioms (e.g. Bernoulli's Inequality)

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Hasse's Local-Global Principle for $\mathbb{Q}$

Some property P is true for $\mathbb{Q}$ iff it is true for all the completions of $\mathbb{Q}$.

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Definition of adele ring for $\mathbb{Q}$

The adele ring $\mathbb{A}_{\mathbb{Q}}$ is defined to be the restricted product of all the completions of $\mathbb{Q}$. Morally speaking, the adele ring can be viewed as the collection of the properties shared by all the completions of $\mathbb{Q}$.

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Idea

Instead of asking whether a property simultaneously holds for all completions of $\mathbb{Q}$ (which forces us to use complicated algebraic constructions like the adele ring $\mathbb{A}_{\mathbb{Q}}$), what if we instead asked if a property holds for the *generic* completion of $\mathbb{Q}$?

Starting point:

For simplicity, let us assume that our base field is $\mathbb{Q}$. Classically, an absolute value of $\mathbb{Q}$ is a function $|\cdot| : \mathbb{Q} \to \mathbb{R}$ such that for all $x, y \in \mathbb{Q}$:

- $|x| \geq 0$, and $|x| = 0$ iff $x = 0$

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We define a *place* as an equivalence class of absolute values whereby $|\cdot|_1 \sim |\cdot|_2$ if there exists some $\alpha \in (0, 1]$ such that $|\cdot|_1 = |\cdot|_2^\alpha$ or $|\cdot|_2 = |\cdot|_1^\alpha$. One might then wish to define the classifying topos of places as the coequaliser of the following diagram in $\mathfrak{Top}$:

$$(0,1] \times [av] \underset{ex}{\overset{\pi}{\rightrightarrows}} [av]$$

Remark #1

The coequaliser of this diagram exists, because it is known that $\mathfrak{Top}$ has all coequalisers. The next task is to describe the points of this topos explicitly, which is a current work-in-progress.
Once we accomplish this, there are many very interesting applications of the idea to current open questions in number theory, model category theory etc.

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Remark #2

The generic model of [places] is not the generic completion. I still need to define the classifying topos of completions with respect to a place. The generic model of *that* classifying topos will be something similar to the original adele ring (cf. [10]).

‘One can hope for a very general method of reduction and ‘dévissage’ that transforms a problem of multiple variables into a problem of a single variable, where the difficulty of the original problem is transformed into a problem of working constructively.’

— André Boileau, and André Joyal

References I

[1] Boileau, A., Joyal, A. *La Logique des Topos*, The Journal of Symbolic Logic, Vol. 46, No. 1 (1981).

[2] Caramello, O., *Theories, Sites, Toposes: Relating and studying mathematical theories through topos-theoretic 'bridges'*, Oxford University Press (2017).

[3] Greenlees, J.P.C.,*Adelic Cohomology*, preprint, `https://arxiv.org/abs/1903.02668`

[4] Johnstone, P.T.: *Topos Theory*, Dover Publications Inc., (1977)

[5] Johnstone, P.T.: *Sketches of an Elephant: A Topos Theory Compendium, Vol. 1*, Clarendon Press, (2002)

[6] Johnstone, P.T.: *Sketches of an Elephant: A Topos Theory Compendium, Vol. 2*, Clarendon Press, (2002)

References II

[7] Mazur, B., *On the Passage from Local to Global in Number Theory*, Bulletin of the AMS, 29 No. 1, (1993).

[8] Maietti, M.E., Vickers, S. *An induction principle for consequence in arithmetic universes*, Journal of Pure and Applied Algebra Volume 216, Issues 8–9, (2012)

[9] Moerdijk, I. *The classifying topos of a continuous groupoid, I.*, Transactions of the American Mathematical Society Volume 310, Number 2, (1988)

[10] Vickers, S., *Localic Completion of Generalized Metric Spaces*, preprint (2003).

[11] Vickers, S., *Continuity and Geometric Logic*, J. Applied Logic (12), 14-27, (2014).

Questions?